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THE CALCULUS FOR ENGINEERS

E.S. ANDREWS AND H.B. HEYWOOD

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THE CALCULUS FOR ENGINEERS

By EWART S. ANDREWS, B.Sc.Eng. (Lond.)

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THE BROADWAY SERIES OF ENGINEERING HANDBOOKS
VOLUME XIII

THE CALCULUS FOR ENGINEERS

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*TWO HUNDRED AND EIGHTY-FOUR PAGES WITH
ONE HUNDRED AND TWO ILLUSTRATIONS
TABLES AND NUMEROUS WORKED
EXAMPLES*

LONDON
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1914

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E.V.

PREFACE

It is generally admitted that the standard courses of pure mathematics as taught in schools and colleges are not well adapted to form part of the professional training of engineers. But mathematics must form a large and an essential part of this training; and there has arisen a large body of text-books written to satisfy the special needs of engineering students. As a rule, however, these text-books will be found to consist of extracts, more or less modified, from the standard courses, interpolated with examples from engineering subjects proper. It is inevitable that the result of this scissors-and-paste method lacks the essential virtues of the original courses, logical development and ease of comprehension. Our experience is that the undoubted difficulty which the engineer has with mathematics is due principally to his general difficulty in reasoning in mathematical language, and that his difficulties are not restricted to the calculus.

In this book, the outcome of the combined efforts of an engineer and a mathematician, it has been our aim and ambition to develop a course *ab initio*, treating engineering calculus as a subject of engineering. This has led us to endeavour to create a whole, logical and consistent in itself, in which the point of view has always been that of the engineer, has always been concrete rather than abstract. We approach each new development of the subject through an engineering problem, so that first the need of a proposition may be felt, and then the proposition provided. Mathematical terms are defined more in every-day language than is usual, and a large amount of consideration is naturally given to typical engineering applications. Particular stress has been laid upon steps of reasoning which experience has shown to present obstacles to students.

The student will find throughout the book a number of exercises by which he may test the extent to which he has followed the various sections of the work. He is strongly recommended to work carefully through these, as it is only by continual practice that he can expect to become fully acquainted with the subject.

The marks (C), (M) and (E) which will be found alongside many of the examples and exercises indicate that the problems are of particular interest to civil and constructional, mechanical and electrical engineers respectively.

For those students who wish to obtain a fair idea of the principles of the calculus as applied to engineering problems, and who have not time to work systematically right through the book, we would recommend the following shortened course: Chapters I and II, IV, V, VI, VII, X.

We wish to express our thanks to Prof. Horace Lamb and Mr. W. W. F. Pullen for permission to include the tables of exponential functions and hyperbolic logarithms respectively, and to Mr. A. E. Monkcom, A.R.C.S., for assistance and suggestions with the proofs.

EWART S. ANDREWS.
H. BRYON HEYWOOD.

LONDON, *August*, 1914.

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SUMMARY OF RESULTS ASSUMED KNOWN.

The following are a few of the more important results that should be known by a student reading this book. They are not intended, however, to represent a complete summary of a preliminary course in Mathematics :—

I. Algebra.

(1) *Representation of quantities by symbols.* (This, however, is developed at length in Chaps. II and III.) Calculation from formulæ.

(2) *The simple rules*—addition, substraction, multiplication and division—and their applications. The student should be able to deal with easy fractions, factorisations, etc.

(3) *Equations*, linear and quadratic, and linear simultaneous equations. Algebraic and graphical solutions of equations.

(4) *Indices.*

$a^{\frac{p}{q}}$ means $\sqrt[q]{a^p}$, the q^{th} root of a^p.

The laws of indices.

$$a^m \times a^n = a^{m+n}$$
$$a^m \div a^n = a^{m-n}$$
$$(a^m)^n = a^{mn}$$
$$a^o = 1.$$

(5) *Logarithms.*

Laws. $\log (a \times b) = \log a + \log b$

$\log (a \div b) = \log a - \log b$

$\log a^m = m \log a.$

$\log_a c = \log_a b \times \log_b c.$

Numerical calculations with logarithms.

II. Geometry.

(1) *Triangles*, especially congruence (*i.e.* equality in all respects) and the following propositions :—

(*a*) The sum of the angles of a triangle is two right angles.

(*b*) *Pythagoras' Theorem*—if c is the hypotenuse and a and b are the sides of a right-angled triangle, then $c^2 = a^2 + b^2$.

(2) *Parallelograms.*

(3) *Circles*—chords, arcs, tangents, and the proposition, If two chords AOB, COD intersect at O, then

$$AO . OB = CO . OD.$$

(4) *Ratio and Proportion*, simple results, especially—

A parallel to the base of a triangle divides the sides proportionally—

And the converse.

Similar figures are figures having the same shape: they can be placed in perspective.

(5) *Mensuration*:—

(*a*) Triangle: area $= \frac{1}{2}bh = \sqrt{s(s-a)(s-b)(s-c)}$
$= \frac{1}{2}ab \sin C$.

(*b*) Trapezium: area $= \frac{1}{2}(a+b)h$.

(c) Circle: circumference $= 2\pi R$
area $= \pi R^2$.

(*d*) Prism (or prismatic cylinder):
volume = height × base
= slant height × right section.

(*e*) Cone: volume = $\frac{1}{3}$ base × height.

(*f*) Sphere: area $= 4\pi R^2$
volume $= \frac{4}{3}\pi R^3$.

(*g*) Spherical Cap: volume $= \frac{1}{3}\pi h^2(3R - h)$.
(h = height of cap).

III. Trigonometry.

(1) The *measurement of angles* in degrees and radians (circular measure).

(2) *Definitions* of the trigonometric ratios:—

$$\sin\theta, \cos\theta, \tan\theta, \operatorname{cosec}\theta, \sec\theta, \cot\theta.$$

(3) *Formulæ.*

$$\cos^2\theta + \sin^2\theta = 1.$$

$$\tan^2\theta = 1 - \sec^2\theta \qquad \cot^2\theta = 1 - \operatorname{cosec}^2\theta.$$

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) = -\cos\left(\frac{\pi}{2} + \theta\right) = \sin(\pi - \theta)$$
$$= \sin(2\pi + \theta) = -\sin(-\theta).$$

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) = \sin\left(\frac{\pi}{2} + \theta\right) = -\cos(\pi - \theta)$$
$$= \cos(2\pi + \theta) = \cos(-\theta).$$

$$\text{Tan } \theta = \cot\left(\frac{\pi}{2} - \theta\right) = -\tan(\pi - \theta).$$

$$\text{Sin } 2\theta = 2 \sin \theta \cos \theta.$$

$$\text{Cos } 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta.$$

$$\text{Sin } (A \pm B) = \sin A \cos \theta \pm \cos A \sin B.$$

$$\text{Cos } (A \pm B) = \cos A \cos B \mp \sin A \sin B.$$

(4) The use of *Trigonometric Tables.* The following special ratios :—

θ	0°	30°	45°	60°	90°
Sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
Cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
Tan θ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Infinite

(5) When θ (measured in radians) is small, sin θ and θ approach each other in value, thus :—

$$\text{Sin } \theta = \theta \quad \text{(approx. for } \theta \text{ small)},$$

and the ratio sin θ/θ approaches unity.

The value of the tangent also approaches the value of θ. These facts are illustrated by the following table :—

Degrees.	Radians (θ).	Sin θ.	Sin θ/θ.	Tan θ.
20°	·3491	·3420	1·0207	·3640
10°	·1745	·1736	1·0049	·1763
5°	·08727	·08716	1·0013	·08749
1°	·0174533	·0174524	1·0001	·0174551

It will be noted that, when the angle is less than 90°,

$$\text{Sin } \theta < \theta < \tan \theta.$$

(6) The following *properties of a triangle*—a, b, c are the sides of the triangle, and A, B, C are the opposite angles :—

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\frac{\text{Sin } A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

THE CALCULUS.

CHAPTER I.

THE CALCULUS—INTRODUCTION AND GRAPHICAL TREATMENT.

§ 1. **Physical Quantities and Numbers.**—The physical quantities with which the engineer mostly has to deal are length, area, volume, time, mass, speed, acceleration, momentum, force, energy or work, power, stress (especially pressure, compression or tension, and shearing stress), and the electrical quantities,—quantity of electricity, resistance, voltage or potential, capacity, electric force, quantity of magnetism, magnetic force, inductance. All these are measurable and can be represented by numbers. It is very important to notice that all measurement is approximate, not exact.

We will take the simplest quantity, length, and what is said of that applies to all. If we use an ordinary scale we can read a length to the nearest $\frac{1}{10}$ in. or by estimation to the nearest $\frac{1}{100}$ in.; if we use a micrometer gauge, however, we can read, say, to the nearest $\frac{1}{1000}$ in., and physicists carry measurement to a much higher degree of accuracy. The reading obtained in the first case would be expressed for instance by 6·43 in., and in the second, 6·427 in.

In any reading the last figure expresses the limit of accuracy *either* of which our instrument is capable *or* which we require for our purpose. It is especially important not to omit final ciphers; thus 6·40 should never be written 6·4, for the latter would imply that we were incapable of reading to hundredths of an inch, so that the reading might really be, for instance, 6·37 or 6·42.

As we have already said, our limit of accuracy may not be the delicacy of our instruments, but the special requirements we have in view. For example, if we were constructing a connecting rod, say 6 ft. long, we should have to make our measurement to the nearest $\frac{1}{100}$ in., but if we were estimating the amount of tar necessary for a path about 6 ft. wide, the nearest inch would be more than accurate enough. This does not mean that we could not measure the path more accurately; doubtless we could measure it to a hundredth of an inch with proper precautions; but the accuracy is sufficient for our purpose.

ERRORS.—Very often we are not quite certain of the last figure of our reading. Such would be the case if we were measuring with a scale graduated to tenths of an inch, and were estimating to hundredths. A reading 6·43 might really be 6·41 or 6·44, say. The discrepancy is called the *error*. In some cases the error may be eliminated or reduced by taking the average of several readings, but it must be borne in mind that taking the average will never give another decimal place; *the average of ten scale readings estimated to hundredths will not give the result correct to a thousandth.*

The error mentioned above is the *absolute error*.

What is usually more important is the *proportional error*, that is the ratio of the absolute error to the reading. The error on the reading 6·41 of the correct value 6·43 is $-\cdot 02$, and the proportional error is $\frac{-\cdot 02}{6\cdot 43} = -\cdot 003$. $\left[\text{It would come to the same thing if we took } \frac{-\cdot 02}{6\cdot 41}.\right]$ Usually the percentage error is taken, i.e. 100 × proportional error, in this case $-\cdot 3$. We have a good example of this kind in the case of **I** beam and like rolled sections used in constructional steelwork. The makers cannot guarantee the sizes of these sections within $2\frac{1}{2}$ per cent. of the standard values. It would therefore be quite useless to calculate the safe load upon a girder as 12·3728 tons, because in that calculation we have assumed that the section is *exactly* as listed, but it is probably not quite, so that 12·4 tons would be all that it is worth while giving as the safe load.

In this connexion we should note that it is always the *significant figures that count, not the decimal point*. If, for instance, we are measuring the extensions in a steel bar 30 in. long, and it comes to ·0176 in. and we want the *strain* (i.e. extension per unit length), we say strain $= \frac{\cdot 0176}{30} = \cdot 000587$; this will then have the same percentage error as our original measurement although there are more places of decimals.

§ 2. **Mathematical Formulæ.**[1] **The Variation and Interdependence of Physical Quantities.**—It is convenient for the sake of brevity to use single letters for physical quantities, and thereby express the relations between them in a compact form. For instance, for a rectangle we have

---

[1] The student is supposed to know a little elementary algebra —see the summary of early work.

$$\text{length} \times \text{breadth} = \text{area},$$

and this may be written

$$l \times b = A, \text{ or simply } lb = A,$$

if we agree that l shall stand for *length*, b for *breadth*, and A for *area*. Or again take the case of a rod of constant cross section stretched by hanging a given weight to its end; the elongation is proportional to the weight and the length of the rod, and is inversely proportional to the cross section. This can all be expressed by the formula

$$\delta = \frac{Wl}{EA} \quad \dots\dots\dots\dots (1)$$

where δ is the elongation, W is the weight applied, l is the length of the rod, A is its cross section, and E is a "numerical constant" (Young's Modulus).

These formulæ express laws connecting physical quantities. To work out a result for any practical case, we must "put into the formula" the numerical values for this case.

VARIABLE QUANTITIES.—So far we have supposed that we had only one isolated case to deal with, for example the elongation of a bar of given dimensions under a given weight. But we have also to think of instances where the physical quantities take a whole range of values. In the cylinder of a steam or gas engine, the volume and pressure vary continuously, and we are concerned with it not in merely one or two special positions but in every position. The volume and the pressure are variable quantities or *variables*. To take some other cases, the temperature of the air is a variable; it varies with[1] the *time*, i.e. from hour to hour and from day to day; the output of an engine varies with the fuel consumed; the electric current in a circuit varies with the voltage. We could consider the example mentioned above (1) as a variable case; we might suppose the weight W to increase continuously, then the elongation δ would vary with the load.

In all these cases we might consider one variable as *independent*, the other as depending on it. The temperature depends on the time in this sense, that if the time is given then the temperature is determined; at 2 o'clock the temperature was 60°, at 2.30 it was 62° and so on (this does not imply a physical relation between temperature and time, but only a correspondence). And similarly with the other cases —the pressure in a cylinder is determined at each position of the piston, so that the pressure is dependent upon the volume. In these two cases respectively the time and volume will be spoken of as *independent variables*, the temperature and pressure as *dependent variables*. But this distinction is largely a matter of convenience: we might quite well, and sometimes do, consider the volume as dependent upon the pressure in the second example, for is not the volume determinate for each given pressure?

EXAMPLES ON VARIABLES.—In most engineering pocket-books will be found the formula for width of shaft keys

---

[1] We must distinguish between the two expressions "varies as" and "varies with". The former implies the proportionality of the two quantities; the latter does not imply any particular manner of varying.

$$b = \tfrac{1}{4}d + \tfrac{1}{8},$$

where b is the breadth of the key in inches, and d is the diameter of the shaft in inches.

For any given value of d we can find b by means of the formula; b is therefore a *dependent variable* because its value depends on the value of d that we are using. We shall have a number of examples of similar formulæ as we proceed in the book.

EXERCISES 1A.

The following exercises are intended to give the student practice in the use of formulæ and the transformation of formulæ from one form into another.

1. The horse-power transmitted by a belt is given by the formula H.P. $= \dfrac{\pi n d\,(T_1 - T_2)}{33000}$, where n is the number of revolutions per minute, d the diameter of the pulley in feet, and T_1 and T_2 the tensions in lb. on the tight and slack sides respectively. If $T_1 = 2\,T_2$ find the tensions when $n = 100$, $d = 4$, and H.P. $= 3\tfrac{1}{2}$.

2. The resistance of electric circuits in parallel is given by the formula $\dfrac{1}{R} = \dfrac{1}{r_1} + \dfrac{1}{r_2} + \ldots$ where R = resistance of whole circuit and r_1, r_2, etc., are the resistances of the separate branches. Find R when $r_1 = 10$ and $r_2 = 20$ ohms in the case of two branches.

3. Using the formula of the previous question, if $r_1 = 15$ ohms, what must r_2 be to make $R = 10$ ohms?

4. Given that 1 in. = 2·54 cm. and 1 kilo. = 2·205 lb., find how many kilos. per sq. cm. correspond to 1 lb per sq. in.

5. A horse-power is measured by 33,000 foot-lb. per minute. If a grm.-centimetre is 981 ergs and 10,000,000 ergs per second make a watt, find how many watts correspond to 1 horse-power. Use the data of question 4.

6. If 42,000,000 ergs are equivalent to the heat required to raise the temperature of 1 grm. of water 1° centigrade, how many foot-lb. will be required to raise 1 lb. of water through 1° Fahr. (a range of 100° C. is equivalent to one of 180° F.)? Use the data of questions 4 and 5.

7. The equation to a parabola is of the form $y = ax^2 + bx + c$. When $x = o$, $y = o$, when $x = \frac{L}{2}$, $y = r$, and when $x = L$, $y = o$; express y in terms of x, L and r.

8. If $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{2 \times 3} + \frac{x^4}{2 \times 3 \times 4} + \ldots$ calculate its value to three decimal places when $x = {\cdot}2$.

9. In the formula $y = ax^n$, $y = 2{\cdot}34$ when $x = 2$, and $y = 20{\cdot}62$ when $x = 5$. Find a and n.

10. The absolute temperatures of a gas before and after adiabatic expansion are given by the formula

$$\frac{T_1}{T_2} = \left(\frac{V_2}{V_1}\right)^{\gamma - 1}.$$

If $\frac{V_1}{V_2} = \frac{1}{6}$ and $\gamma = 1{\cdot}408$, find T_2 if $T_1 = 531^\circ$ F. (abs.).

11. The mean pressure in a steam engine cylinder under isothermal expansion is given by the formula $p_m = \frac{p_1}{r}(1 + 2{\cdot}3 \log_{10} r)$, where p_1 is the initial pressure and r is the ratio of expansion. Find p_m if $p_1 = 120$ lb. per sq. in., and $r = 6$.

12. The flow of water in cubic feet per second over a triangular notch, the angle of which is a right angle is given by the formula $Q = \frac{8}{15} C_d \sqrt{2g}\, H^{\frac{5}{2}}$,

where C_a is the "coefficient of discharge" and H is the head in feet. Find the flow if $C_a = \cdot 60$ and the head is $15\frac{1}{2}$ in.

13. If $e = 2\cdot 718$, find $e^{\frac{1}{4}}$.

14. If $y = 3 \sin x + 4 \cos x$, find y for $x = 120°$.

15. The ratio of the tensions T_1 and T_2 on the tight and slack sides of the belt running over a pulley is given by the formula: $2\cdot 3 \log_{10} \frac{T_1}{T_2} = \mu\theta$, where μ is the coefficient of friction and θ is the angle of lap in radians of the belt upon the pulley. If $T_1 = 480$, $\mu = \cdot 4$, and $\theta = 120°$, find T_2.

16. The efficiency of a series of n pulleys is equal to η^n where η is the efficiency of each pulley. If η is $\cdot 94$, find the least number of pulleys that must be employed if the efficiency of the series is less than 50 per cent.

17. The horse-power absorbed in friction in a frusto-conical pivot is $\frac{2\pi NT}{33000}$ where

$$T = \frac{2\mu W (R^3 - R_1{}^3)}{12 \times 3 \sin \alpha (R^2 - R_1{}^2)}.$$

Find the horse-power if $N = 140$, $R = 2$, $R_1 = \cdot 75$, $\mu = \cdot 08$, $\alpha = 30°$ and $W = 2500$.

18. 50 cells the voltage of each of which is 1·8 volts are connected in series and the circuit is completed by a wire of 15 ohms resistance. If each cell has an internal resistance of ·3 ohms, calculate the current in the circuit by the formula

$$C \text{ (amperes)} = \frac{E \text{ (volts)}}{R \text{ (ohms)}}$$

where E = total voltage and R = total resistance.

19. Using the result of question 5, find the horse-power absorbed by an arc lamp using 9 amperes and having a voltage drop of 50 volts by the formula

$$\text{Watts} = \text{amperes} \times \text{voltage drop}.$$

20. Hutton's formula for wind pressure on inclined surfaces is

$$P_\theta = P_\nu \sin \theta^{1\cdot84 \cos\theta - 1},$$

where P_ν = pressure on a vertical surface, and θ is the angle of inclination of the surface to the horizontal. Find P_θ if $P_\nu = 50$, and $\theta = 30°$.

§ 3. **The Expression of Physical Relations by means of Graphs.**—The representation of the dependence of two variable quantities by means of a curve will already be familiar to the student, for even if he has not done class work in "graphs," he will have seen examples daily—the weather charts in the papers, the diagrams showing growth of population, of exports and imports, and so forth. Let us take a weather chart as a first example.

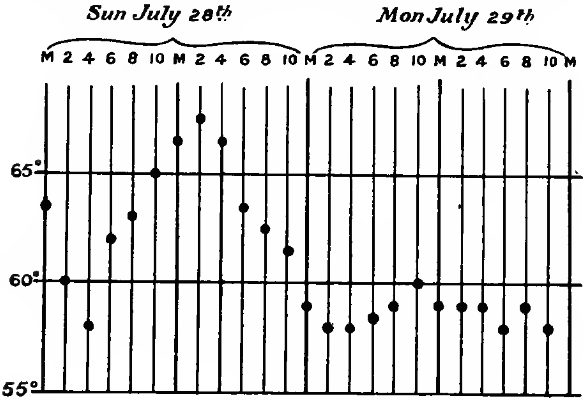

Fig. 1.—The above chart is constructed to show the thermometric records for two days, the dot indicating the position of the thermometer every two hours. From "The Daily Telegraph," 30 July, 1912.

In this chart observations have been taken at intervals of two hours. We cannot get an exact temperature curve from this, as we do not know the temperature between each pair of observations. However if *we draw a smooth curve* through the dots,[1] we shall obtain a curve which differs very slightly from the true one. In making this statement we rely on the fact that the temperature does not vary in a sudden and irregular manner in intervals so short as two hours: from 6 a.m. to 10 a.m. on Sunday the temperature was rising; it was 63° at 8 a.m. and 65° at 10 a.m.; at 9 a.m. no doubt it was not very far from 64°.

This process is called *interpolating*: a good deal of judgment must be exercised in doing it. The

---

[1] The art of drawing a good smooth curve through a number of points is very valuable to an engineer and one rather difficult to acquire. It will assist to remember that the piece of the curve which lies between two consecutive points is affected by other points on either side. This is seen in the diagram. Before drawing in a piece of the curve freehand, the pen or pencil should be run two or three times just over the points in order to accustom the hand to the shape. A "French curve" may also be used.

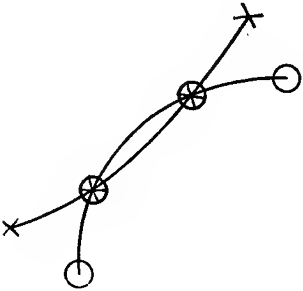

FIG. 2.

nature of the data given, and the nature of the graph required must be taken into account. For instance in some cases the data are known to be not very exact while the graph is known to be smooth (as in a "stress-strain" curve for a metal bar, for instance); the graph need not then be drawn through the actual points, but should strike a smooth medium path amongst them. In other cases, although the data are exact, we do not require the exact graph with all its irregularities but something to show us the general tendency of the dependent variable. Such modification of the data is dangerous, however; in the present instance we have relied implicitly on the "Daily Telegraph" and have drawn the curve strictly through the dots.

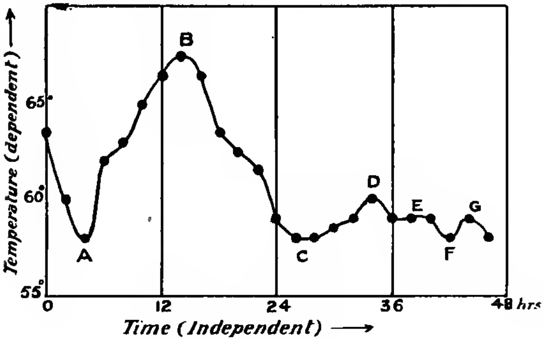

FIG. 3.

SLOPES AND RATES.—Notice that *when the curve slopes upwards the temperature is increasing*, and *the steeper the slope, the more rapid is the rate of rise.* When the curve slopes downwards the temperature is diminishing.

CRITICAL VALUES.—When the curve is horizontal the temperature is neither rising nor falling, i.e. it is *stationary*. Such points occur at ABCDEFG (Fig. 3); the temperature here is said to take *critical values*.

At B, D, G the temperature has *maximum values*: they are preceded by a *rise* and followed by *a fall*.

At A, C, F the temperature has *minimum values*: these are preceded by a *fall* and followed by a *rise*.

At E is a *point of inflexion*: it is preceded and followed by a fall. A point of inflexion clearly may also be preceded and followed by a rise. At a point of inflexion the curve changes from concave to convex or vice versa.

These ideas are of the utmost importance, being at the basis of the calculus: we shall use a more convenient curve for their numerical discussion.

§4.—**Slopes.**[1] MEASUREMENT OF SLOPES AND RATES; EXAMPLE OF STEAM-ENGINE MECHANISM.— In the following curve, the *abscissæ* (i.e. distances measured along the horizontal *axis* OX) represent the time, while the *ordinates* (i.e. distances measured vertically from OX) represent the displacements of a point (the cross-head in the steam-engine mechanism, see p. 93). The rate of increase of the ordinate measures the movement of the cross-head in a unit of time and so is the speed of the point.

Take any point M on the axis OX, and let MP be the corresponding ordinate. Let us represent the length OM by the letter t, and the length MP by y.

---

[1] *Slopes* or *Rates* are called also *Differential Coefficients* and *Derivatives*. These words all mean what amounts to the same thing but in different connexions. We shall later on use the term *Differential Coefficent* as it is the one in most general use.

Then t is the time which elapses from the starting-point, and y is the displacement. Take now another ordinate NQ: MN is the increase or *increment* in the time, and if we draw PK horizontally, KQ is the increment in the displacement. We shall use the symbol δ to denote "*a change in*" and then we may write

$$MN = \delta t; \qquad KQ = \delta y.$$[1]

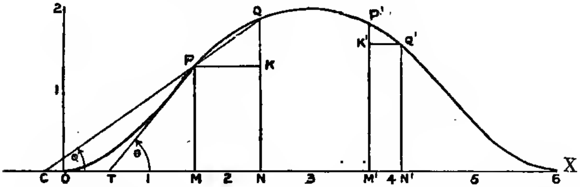

FIG. 4.—Displacement curve for steam engine mechanism: connecting rod $3\frac{1}{2}$ × crank. Ordinate = displacement of cross head: I unit = crank length. Abscissa = time or angle through which crank has turned: 1 unit = 60°.

Now as the average rate of increase of a quantity is equal to the total increase divided by the time, we have:—

$$\text{average rate of increase of } y = \frac{\delta y}{\delta t}$$

$$\doteq \text{average speed of cross-head.}$$

If the chord PQ meets the axis OX in C, and if we write $M\hat{C}P = \phi$ = angle the chord PQ makes with the axis of x, then we have $K\hat{P}Q = M\hat{C}P = \phi$, since PK is parallel to CM.

Hence average rate of increase of y

---

[1] Note very carefully that δ is not a symbol denoting quantity: it is more of the nature of the signs +, ×, etc. δy is itself one single quantity, the increment in y.

$$=\frac{\delta y}{\delta t} = \frac{\mathrm{KQ}}{\mathrm{PK}} = \tan\phi.^1$$

A glance at the figure shows that the average rate of increase is less than the actual rate at the point P (where the curve is steeper). Suppose now that the ordinate NQ moves up towards MP. It is evident that *the average rate approximates to the actual rate at P.* And if Q were sufficiently near to P, the average rate and the true rate at P could be considered as identical. At the same time the chord CPQ would become indistinguishable from the *tangent at P* to the curve. Calling θ the inclination $\mathrm{M\hat{T}P}$ of the tangent at P to the axis of x, we must therefore have

True rate of increase at P $=$ *tan* $\theta = \frac{\delta y}{\delta t}$ (when δt and with it δy become very small).

This we define as the *slope of the curve* at P and we use for it the notation

$$\frac{dy}{dt}.$$

This slope is in essence a ratio but it is best for most purposes to regard $\frac{dy}{dt}$ as being *in one piece*, the parts dy and dt not having any separate meaning, while $\frac{\delta y}{\delta t}$ on the other hand is a fraction pure and simple.

We are thus able to find a rate from a graph—in this case, a speed from a displacement curve. We

---

[1] See summary of early work for the meaning of tan ϕ.

have only to draw the tangent to the curve at the point in question and find its slope.[1]

Negative Rates.—The right hand half of the curve descends, or slopes downwards. If we take an ordinate M′P′ and a consecutive ordinate N′Q′ in the same way as before, the second ordinate is shorter than the first. *The increment or change in the ordinate is a negative quantity,* and is equal to *minus* K′P′, i.e.

$$\delta y = -\text{K}'\text{P}'.$$

The rate or slope is obtained as before, and it is still $\tan\theta'$ or $\frac{dy}{dx}$, but it has a negative value; the angle θ' is the angle which the tangent at P′ makes with the axis of x and is, according to the usual convention, considered negative.

Zero Rates.—At the summit of the curve and also at O and x, the slope, or rate of increase is zero,

---

[1] It is not easy to draw a tangent to a curve at a point *accurately.* A good plan is to mark off two near points K, L (Fig. 5), on the curve equidistant from P. Then KL gives a better direction than an attempted tangent. The slope may be found conveniently by finding the rise between two points on the line on ordinates (say) 10 units apart. The slope need not always be measured numerically, but the length representing the slope can be picked off with dividers.

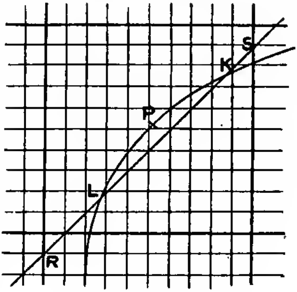

Fig. 5.

$$\text{i.e.} \quad \frac{dy}{dx} = 0.$$

This clearly holds at all critical points.

§ 5. **Derived Curves.**—We can find the rate at any point of the curve in the above section: the rate then is a variable quantity which takes up one value at each value of the abscissa, and therefore it can be graphed in a similar diagram. Such a curve is called a derived curve (Fig. 6). In this case it is a *curve of speeds*—a velocity curve.

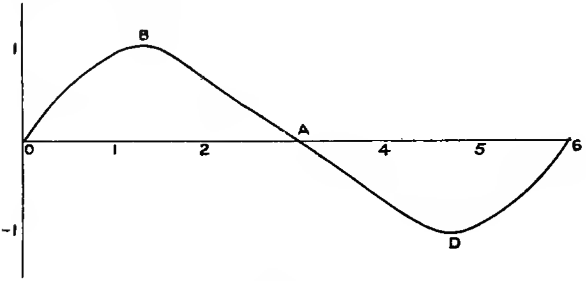

FIG. 6.

It will be observed in Fig. 6 that the slope is initially zero, that it is most rapid in the neighbourhood of B, then diminishes down to zero at A. It then becomes negative, the greatest negative value occurring at B and finally the slope vanishes again at 6. Note that the derived curve crosses the axis at the critical points of the original curve.

The derived curve is obtained graphically by finding the slope at a series of points on the original curve, and so obtaining a series of points on the new curve. It is one of the most unsatisfactory processes in graphics, but (when the law is not given by a formula so that the methods of Chapter III do not apply) there is no other method of attaining the result.

SECOND DERIVED CURVES.—Starting with the derived curve we can find *its* slope at each point, and so obtain its derived curve. The second derived curve will express the rate of change of the rate of change; in the case in point it will express the rate of change of the speed and will therefore be the *acceleration curve* (Fig. 7).

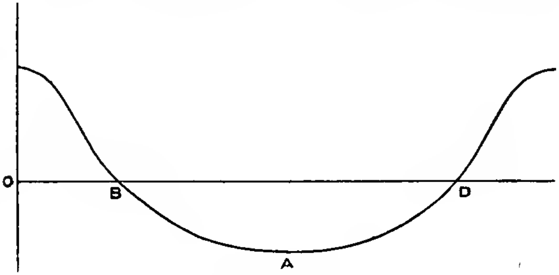

FIG. 7.

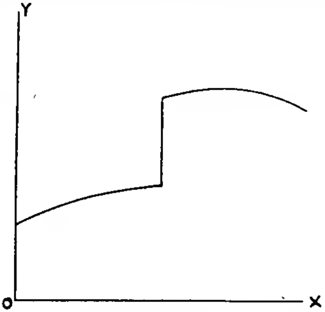

FIG. 8.

Note.—Curves are sometimes met with which make a sudden jump, as in the figure. Such curves are called *discontinuous*: they have of course no "slope" at the points of discontinuity.

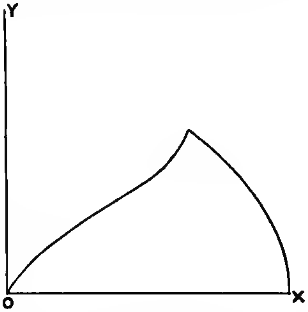

FIG. 9.

More frequently we find curves with sharp angles. In these the slope makes a sudden change at the angles, and is discontinuous, but the curve itself is continuous.

If a curve is discontinuous or if its derived curve is discontinuous we cannot express the quantities which it represents by a general formula.

EXAMPLES OF DISCONTINUITY.

Shear and Bending Moment Curves.—A very familiar example of a discontinuous curve in engineering work occurs in the case of shear diagrams for beams with isolated load systems. Fig. 10 shows such a case.

Now it can be shown that the shear diagram for a beam is the derived or slope curve for the bending moment diagram and that the load diagram is the derived curve for the shear curve. It is clear from the figure that the bending moment diagram has sharp angles so that we see that when a curve is discontinuous, it is derived from a curve with sharp angles. Further, the derived curve for a curve with sharp angles will consist of a number of isolated pieces; this is clear from the load curve which consists of a number of isolated forces or vertical lines.

Indicator Diagrams.—In theoretical indicator diagrams for showing the variation of pressure at different points in the stroke of a steam engine (Fig. 11), we have another familiar example of a curve with sharp angles; these points occur at ABCDE which represent the "cut off," "release," "exhaust," "compression" and "full admission" respectively.

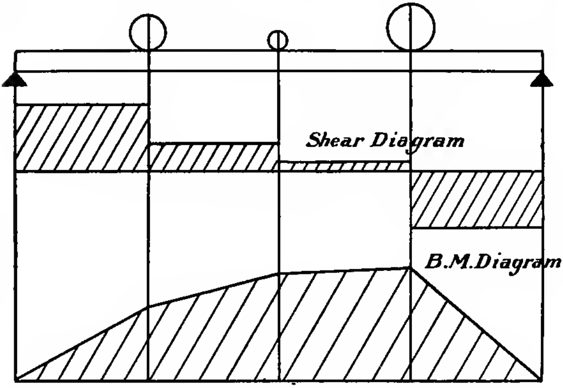

FIG. 10.

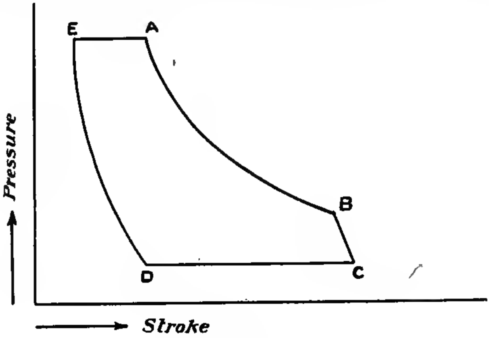

FIG. 11.

§ 6. **Sum Curves.**—The following curve is a *crank-effort or force diagram.* We use it to illustrate curves of areas, *sum or integral curves.*

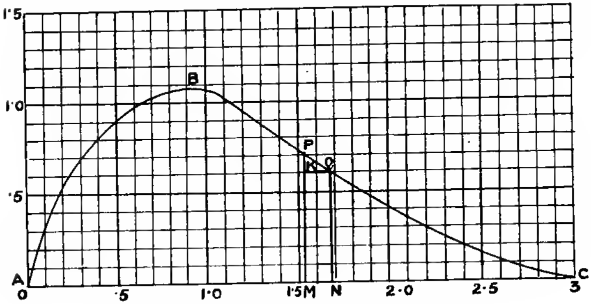

Fig. 12.—Crank-effort Diagram.

Table of Areas.

x =	·0	·2	·4	·6	·8	1·0	1·2	1·4	1·6	1·8	2·0	2·2	2·4	2·6	2·8	3·0
areas =	·0	·05	·19	·37	·57	·79	·99	1·17	1·31	1·43	1·53	1·60	1·65	1·68	1·70	1·70

In this diagram the abscissæ represent the distance through which the crank pin moves up to a half revolution; the ordinate represents the crank-effort, a force acting upon the crank. Take any ordinate MP, and an ordinate NQ very near to it. When the crank pin moves through the distance MN, the piston will do work equal to

$$\text{average effort} \times \text{MN}.$$

Now MP and NQ are very nearly equal, so that the work done is approximately

$$\text{NQ} \times \text{MN} = \text{rectangle NK}.$$

The rectangle is practically a slice of the curve, if we neglect the triangle at the top of the rectangle. So that the *work done is equal to the area of the curve between the two corresponding ordinates.*

This applies to the motion through a very short distance MN; if we want the work done by the piston while the crank pin moves from A to M, we shall only have to add together all the similar slices of the curve for a number of short motions making up AM. Thus the work done by the piston while the crank pin moves from A to M is equal to the area under the curve from A to M. This is an *exact* result, for we can make the short motions like MN as small as we please, and hence the row of triangles like QPK will form a vanishing area lying along the curve: in fact these triangles occur only in the proof and not in the result.

We can now draw a second curve the *sum*[1] curve (Fig. 13), in which the abscissæ are the same as before, and the ordinate is *the area of the original or primitive curve up to the corresponding point.*

The student will notice that the sum curve must start at zero at A and rise continuously; also that it will move rapidly when the ordinate of the orginal curve is longer, so that *area* is being added on faster, in fact its rate of rise or *its slope is proportional to the ordinate of the original curve.* This fact is very important and we shall return to it.

We have described this process for a special curve —crank-effort or force diagram—but it may evidently be applied to any curve, whatever its significance.

---

[1] The reason for calling this a sum curve is apparent already, and will be explained later.

As a second example we will take a velocity curve. In the following diagram the abscissa represents the time, and the ordinate is the velocity of a point at that time.

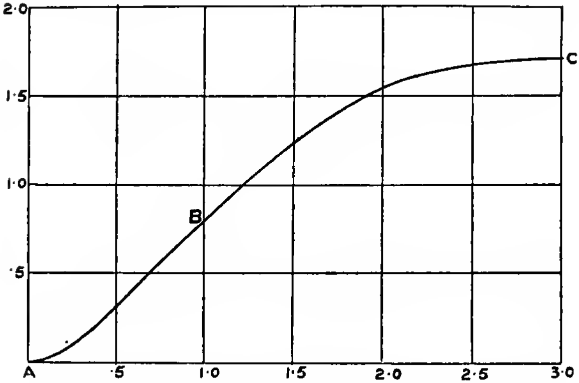

Fig. 13.—Work Curve Diagram showing Work Done.

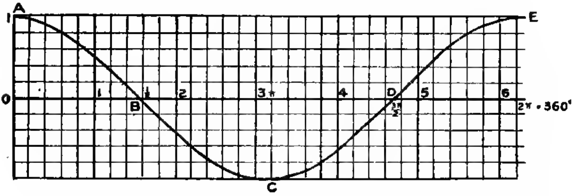

Fig. 14.—Velocity Curve. [This curve is the cosine curve, and represents the velocity in a harmonic motion.]

Now the distance covered by a moving point is the average velocity multiplied by the time: for a very short period of time the variation of the velocity may be neglected and the short distance described is this velocity multiplied by the short interval of time. The student will see that if we measure from the starting-point, *the displacement of the moving point is given by the sum curve*, which is therefore the displacement curve.

It will be observed that so long as the velocity curve is positive, the displacement curve rises. At B the ordinate of the velocity curve becomes negative, and with it the added area. (Areas below the axis are negative.) The moving point comes to rest at B and begins to move backwards. So the sum curve ceases to rise and begins to fall; it continues to fall so long as the velocity is negative. It will be remarked that after C, the *total area* of the velocity curve is negative, since the negative area below the axis exceeds the positive area above it, and consequently the sum curve is negative.

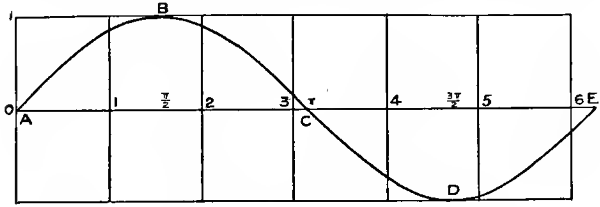

FIG. 15.—Displacement Curve.

§7.—**Connection between sum curves and derived curves.** We shall now prove, what the student has already suspected, that *if one curve is the derived curve of another, then the second is the sum curve of the first.* We shall use a pair of curves that we have already discussed, namely Figs. 4 and 6 which we reproduce (Figs. 16 and 17).

The ordinate LR in Fig. 17 is approximately equal to $\frac{\text{KQ}}{\text{PK}} = \frac{\delta y}{\delta x}$ in Fig. 16 (in the system of units adopted). Calling LR = z, we have then

$$z = \frac{\delta y}{\delta x} \text{ (approximately).}$$

And we have to prove that the increment in the ordinate in Fig. 16 is equal to an increment of area in Fig. 17. That is to say we must show that

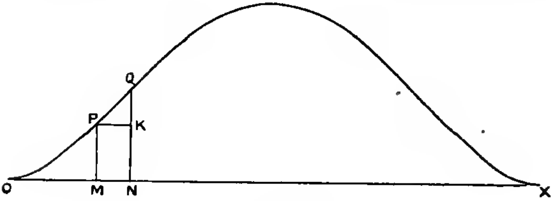

FIG. 16.

$$\delta y = z\delta x \text{ (approximately),}$$

and this is clearly true.

This result is, of course, not rigidly proved, since each of the equations is only an approximation; however we have seen already that the shorter the interval considered is, the closer do these approximations become, and, by taking the intervals short enough, we can make the approximations as close as we please. Moreover we have also seen that all practical work is approximate in nature and not exact (§ 1, p. 1). Hence the mathematical result covers

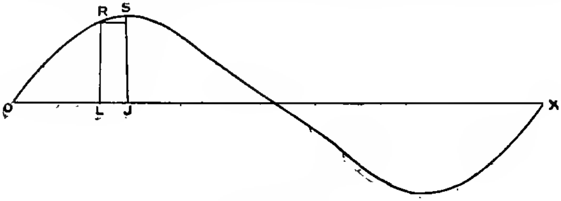

FIG. 17.

all that we do, or ever could, require. The mathematical result is *exact*, but to prove this rigidly, it would be necessary to introduce mental processes[1] with which the engineering student is not, as a rule, familiar.[2]

§ 8.—**Measurement of Areas.**—FIRST METHOD —COUNTING SQUARES.—To obtain the sum curve, the simplest method is to draw the original curve on squared paper, and to count the squares. Fig. 13 has been obtained from Fig. 12 in this way.

The determination of areas is very important, and we proceed to give several other methods: these methods are more accurate than the one already given, but (excepting the second) they are not so well adapted for finding a growing area at each stage. We shall meet with the method of *calculating* areas in Chapter V when we are doing symbolic integration.

SECOND METHOD.—The sum curve can be obtained graphically as follows. Let ACD, Fig. 18, be any primitive curve on a straight base AB. Divide AB into any number of parts, not necessarily equal (but for convenience of working they are generally taken as equal). These so-called base elements should be taken so small that the portion of the curve above them may be taken as a straight line. About 1 cm. or ·4 in. will usually be a suitable size and in most cases a smaller element 11 will come at the end. Find the mid-points, 1, 2, 3, etc., of each of the base elements and let the verticals through these mid-points meet the curve in 1*a*, 2*a*, 3*a*, etc. Now project the last points on a vertical line AE, thus obtaining the points 1*b*, 2*b*, 3*b*, etc., and join such points to a pole P on AB produced and at some convenient distance *p* from A. Across space 1 then draw A*d* parallel to P 1*b*; *de* across space 2 parallel to P 2*b*, and so on, until the point *n* is reached. Then the curve A *de* . . . *n* is the sum curve of the given

---

[1] Especially the method of Limits: this we shall, as a matter of fact, employ later on, but not in a rigorous form.

[2] Obtaining a sum curve is called *integrating*, the inverse process of obtaining a derived curve is *differentiating*: see later (p. 79).

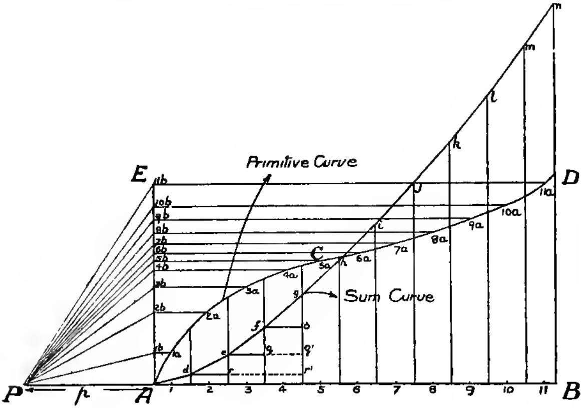

FIG. 18.—Sum-curve Construction.

curve, and to some scale B*n* represents the area of the whole curve.

PROOF.—Consider one of the elements, say 4, and draw *fo* horizontally.

Now $\Delta\, f\, g\, o$ is similar to the Δ P 4*b* A

$$\therefore \frac{go}{fo} = \frac{4b\ \text{A}}{\text{PA}}.$$

but PA $= p$ and 4*b*, A $=$ 4, 4*a*

$$\therefore go = \frac{fo \times 4,\, 4a}{p} = \frac{\text{area of element 4 of curve}}{p}.$$

Similarly $fq = \dfrac{\text{area of element 3 of curve}}{p}$ and so on

$$\therefore \text{Ordinate through } g = go + fq + \ldots$$

$$= \frac{\text{area of first four elements of curve}}{p}$$

∴ The curve A de . . . h is the sum curve required.

Then if B n be measured on the vertical scale and p be measured on the horizontal scale, the area of the whole curve will be equal to $p \times \mathrm{B}n$.

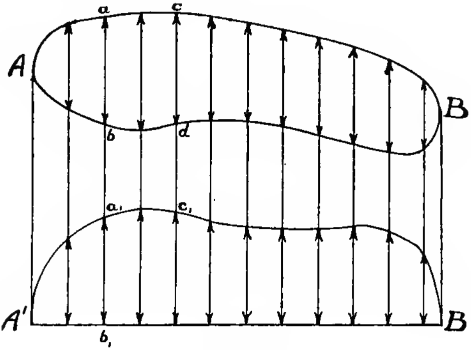

Fig. 19.

It is obviously advisable to make p some convenient round number of units.

The sum curve obtained by this method may have the same operation performed on it, and thus the second sum curve of the primitive curve is obtained, and so on.

If the operation be performed on a rectangle, the sum curve will obviously come a sloping straight line, and if the sum curve of a sloping straight line be drawn, it will be found to be a parabola (§ 12). In the case in which it is required to apply this construction to a curve which is not on a straight base, the curve is first brought to a straight base as follows :—

Suppose A*c*B*d*, Fig. 19, is a closed curve. Draw verticals through AB to meet a horizontal base AB. Divide the curve into a number of segments by vertical lines at short distances apart, and set up from the base AB lengths a_1b_1, etc., equal to the vertical portions *ab*, etc., on the curve. Joining up the points thus obtained we get the corresponding curve $A'a_1c_1B'$, on a straight base.

THIRD METHOD. PLANIMETER.—This instrument is to be found in most drawing offices, and generally provides the quickest and most accurate method of measuring areas. The pattern in most general use is Amsler's.

A square rod A, Fig. 20, carries at its end a pointer E and has adjustably fixed to it a slide B to which is pivoted a second rod C. This rod carries at its end a weighted pin D which is pressed into the paper and thus fixed relatively thereto. The pointer E is first placed at some point X on the boundary of the area and the reading upon the graduated wheel F which runs over the paper is taken by aid of a vernier. The pointer is then moved carefully round the curve until the point X is again reached. The reading of the wheel is then taken again and the difference between the two readings gives the area of the curve. The measuring wheel F is geared up to a second wheel G. The scale to which the areas are read is adjusted by moving the slide B along the rod A. Students should endeavour to obtain practice in the use of this instrument.

The proof of the working of the instrument involves some rather advanced work which is beyond

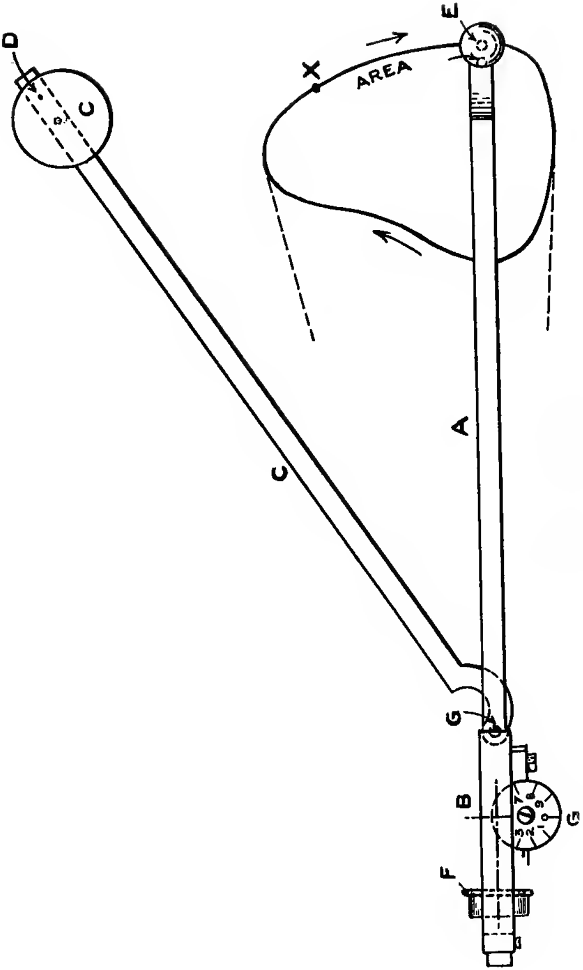

Fig. 20.—Planimeter.

our present scope and it is rather better to accept its accuracy without proof than to attempt a proof that is not really sound.

FOURTH METHOD. MID-ORDINATE RULE.—The area of a curve is the average height (mean ordinate) × the base.

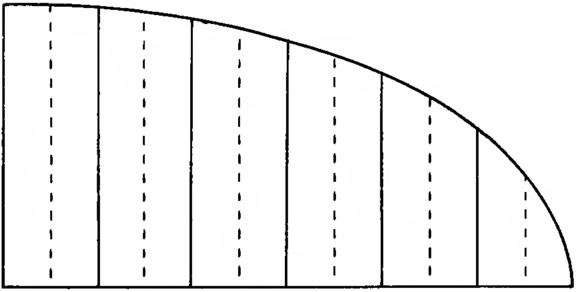

FIG. 21.

Subdivide the curve into vertical strips of equal breadth, and draw the *mid-ordinate* of each strip (the dotted lines). The average height of the curve is approximately the average of these dotted lines. The greater the number of strips the closer will the approximation be. If the length of the base is a, and there are n strips with mid-ordinates $= y_1, y_2, \ldots, y_n$, then we shall have

$$\text{Area} = a \times \frac{y_1 + y_2 + \ldots + y_n}{n}$$

$$= \frac{a}{n}(y_1 + y_2 + \ldots + y_n).$$

The curve in the figure is a quarter of an ellipse. We have chosen intentionally a curve which becomes steep, and which therefore is not well adapted to Simpson's and some of the other rules—(see the tests applied to this curve below).

Fifth Method. Simpson's Rule.—The area must be subdivided into an *even* number of strips (in this case 6). If the *seven* ordinates which bound these *six* strips are called $y_1, y_2, y_3, y_4, y_5, y_6, y_7$, and the *breadth of each strip* is δ, then

$$\text{Area} = \frac{\delta}{3}[y_1 + y_7 + 4(y_2 + y_4 + y_6) + 2(y_3 + y_5)]$$

$$= \frac{\delta}{3}[\text{sum of extreme ordinates, plus four times}$$

sum of even ordinates, plus twice sum of odd ordinates].

This is Simpson's *First* Rule: we do not think it worth while increasing the number of methods unduly by giving Simpson's Second Rule.

Sixth Method. Weddle's Rule.—This is the most accurate of this type of methods. The only objection to it is that the area must be subdivided into six equal parts, or a multiple of six and not any other number. Using the same notation as before the rule is

$$\text{Area} = \frac{3\delta}{10}[y_1 + y_3 + y_5 + y_7 + 5(y_2 + y_6) + 6y_4].$$

The following are the determinations of the area of the above curve by the several methods:—

		Error.	Percentage Error.
Exact value	14·137 . . .	nil.	—
1. Counting squares	14·09	− 0·05	− 0·3
2.	14·2	+ 0·1	+ 0·7
3. Planimeter (three readings)	14·13	0·01	0·1
	14·11	− 0·03	− 0·2
	14·16	+ 0·02	+ 0·2
Mean	14·13	0·01	0·1
4 Mid ordinates	14·26	+ 0·12	+ 0·9
5. Simpson's Rule	14·00	− 0·14	1·0
6. Weddle's Rule	14·03	− 0·11	− 0·8

Note that on account of the "steepness" results for rules 4, 5, 6 are not very good: usually Weddle's ruleshows a more marked superiority over the others than it does here.

Remarks on the last three rules.—Greater accuracy is always obtained by increasing the number of strips into which the curve is subdivided. These rules give the best results for curves which are smooth and which do not become *steep.*

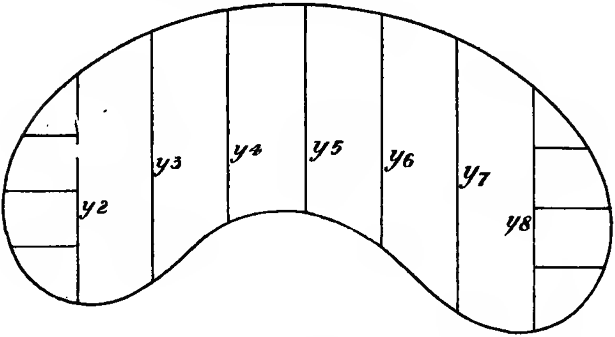

FIG. 22.

They apply equally well to *closed* curves (Fig. 22) which must be divided into strips of equal breadth as before. In the figure there are eight strips, and the end ordinates y_1 and y_9 both vanish. But at the ends the curve is *steep*, so it is better to separate the end strips and cut them up in the perpendicular direction.

The rules (especially Weddle's) will even apply to irregular areas in which the ordinates meet the bounding curve more than twice. In Fig. 23 the fourth ordinate LMNP meets the curve four times.

The corresponding length y_3 is the *sum of LM and NP*. Great accuracy must not be expected as the curve becomes steep several times, but the value obtained will be good enough for most purposes.

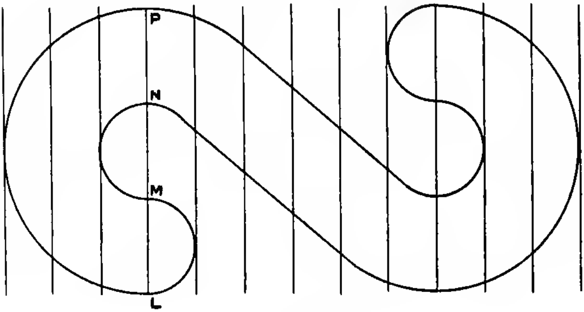

FIG. 23.

EXERCISES 1B.

1. x and t are the distances in miles and the time in hours of a train from a railway terminus. The slope of this curve at any point is a measure of the speed. What is the greatest speed and at what point does it occur?

x	0	1·5	6·0	14·0	19·0	21·0	21·5	21·8	23·0	24·7	26·8
t	0	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0

2. The following values of p and θ are given. Find the slope $\frac{dp}{d\theta}$ when $\theta = 115$.

θ	p
100	14·70
105	17·53
110	20·80
115	24·54
120	28·83
125	33·71
130	39·25

3. The following formula gives the increase c in volume of one pound of water in changing from water to steam at an absolute temperature t centigrade and pressure p lb. per sq. ft., L being the latent heat and J the mechanical equivalent of heat.

$$c = \frac{\mathrm{JL}}{t} \cdot \frac{dt}{dp}$$

$\frac{dt}{dp}$ is thus the slope of the curve of t plotted against p. Find c at $t = 428$ given $\mathrm{L} = 497{\cdot}2$ and $\mathrm{J} = 1393$, the relation between t and p being as follows:—

t	p
413	7563
418	8698
423	9966
428	11380
433	12940
438	14680
443	16580

4. The following numbers give the speed v of a train in miles per hour and the time t in minutes from starting. Draw the sum curve which gives the distances passed through and find the total distance.

v	0	2·4	4·7	7·2	9·6	12·0	14·3	16·9	18·9	20·7	22·2	23·4	24·3	24·9
t	0·00	0·04	0·08	0·12	0·16	0·20	0·24	0·28	0·32	0·36	0·40	0·44	0·48	0·52

5. The area A square feet of the horizontal section of a reservoir at a height h from the lowest point is as follows :—

h	0	2·5	4	7	10	12·5	15	17·5	20	23	25	28	30
A	0	2510	3400	4520	5160	5490	5810	6210	6890	7810	8270	8679	8780

Plot A against h and draw the sum curve, the ordinates of which will be a measure of the volume of water contained up to each height. What is the total volume and the change in volume if the height drops from 15·5 to 14·5?

6. The following figures give the wind pressure p in lb. per sq. ft. on a vertical plane surface at heights h in feet above the ground.

h	5	15	25	35	50
p	13	22	24	27	31

What is the total wind force on a wall 100 ft. long and 40 ft. high?

7. An area is divided into ten equal parts by 11 equidistant parallel lines ·2 in. apart, the first and last touching the bounding curve. The lengths in inches of these ordinates are: 0, 1·24, 2·37, 4·10, 5·28, 4·76, 4·60, 4·36, 2·45, 1·62, 0. Find by Simpson's rule the area in square inches.

8. A circle is drawn of 8 in. diameter. The diameter is divided into 8 equal parts and ordinates are

drawn at right angles to the diameter. Find the area by Simpson's rule, checking the ordinates by calculation for preference. What is the percentage error?

9. x being the distance in feet across a river 80 ft. wide, measuring from one side and y the depth of water in feet, the following measurements are made:—

x	0	10	25	33	40	48	60	70
y	0	4	7	8	10	9	6	4

Find the area of the cross section. If the average speed of the water normal to the section is 3·2 ft. per sec. find the flow in cub. ft. per sec.

10. In a determination of the area of a field, x is distance measured along a straight line AB from the point A; the values of y are offsets or distances, both in chains measured at right angles to AB to the border of the field. Find the average breadth of the field from the line AB to the border.

x	0	1·50	3·00	5·00	7·50	9·00
y	0·53	0·47	0·40	0·42	0·46	0·52

ANSWERS TO EXERCISES.

1A.

1. 183·2 and 91·6 lb. 2. 6·67 ohms. 3. 30 ohms. 4. ·07 kilo. per sq. cm. 5. 746 watts. 6. 778 foot-lb. 7. $y = \frac{4\ xr}{L^2}(L - x)$. 8. 1·221. 9. $a = 0{\cdot}4514$, $n = 2{\cdot}374$. 10. 255° F. absolute.

11. 55·8 lb. per sq. in. **12.** 4·85 cub. ft. per second. **13.** 1·284. **14.** ·598. **15.** 207·7 lb. **16.** 12. **17.** 1·32. **18.** 3 amperes. **19.** ·603 H.P. **20.** 33 lb. per sq. in.

1B.

1. 80 miles an hour at 6 miles. **2.** ·80. **3.** 5·45. **4.** 7·56 miles. **5.** 174800 cub. ft.; 5810 cub. ft. **6.** 85200 lb. **7.** 6·25 sq. in. **8.** 49·33 sq. in.; 1·89 per cent. **9.** 449 sq. ft. 1437 cub. ft. per second. **10.** ·45 chains.

CHAPTER II.

LAWS EXPRESSED SYMBOLICALLY.[1]

§ 9. **Formulæ.**—To know the law connecting two physical quantities, x and y, say, where y is supposed dependent, we must know the value of y corresponding to every value of x. The law can always be expressed by a graph. But very often, especially when it is obtained theoretically, it can be put into a more definite form. For example, when two rough bodies are in contact, we know that the limiting friction between them is *proportional to the pressure*, a law which we express

$$\mathrm{F} = \mu \mathrm{P}.$$

Or again, for a mass of perfect gas at constant temperature, we know that the volume is *inversely proportional to the pressure* (Boyle's or Mariotte's Law), and this we express

$$p = \frac{k}{v}, \text{ or } pv = k.$$

Whenever a law, experimental or theoretical, can be put into a definite shape in this way, it can always be condensed into a *formula*. In fact most laws are too complicated to be expressed in words at all, and then a formula is our only resource besides a graph. Even when the law cannot be put exactly into this definite form, and as a matter of fact no experimental law can be exact, we can still find a formula which expresses a law approximating very closely to it. Such a law devised to fit a series of experimental data is called *empirical.* There is a great advantage in finding an empirical law, for not only does it give us in a very brief and convenient form what could otherwise only be expressed by a complicated table of values, or by a graph, but it often throws a great deal of light on the nature and relations of the quantities involved, and leads to new series of laws and results.

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[1] This and the next chapter may be omitted by students who have already a knowledge of graphs and the laws which they represent.

The student will notice that two important processes are involved. The first is to draw a graph from a formula ; the second is, given a graph, to deduce an empirical formula.

We shall now give a list of the simplest formulæ which occur, including nearly all those that an engineer is likely to encounter, and discuss them briefly.

I. ALGEBRAICAL FORMULÆ.

§ 10. **Algebraical Laws.**—These are the laws which involve only the processes of addition, subtraction, multiplication, division, and extraction of roots.

We remind the student that there are two square roots of any number x, which are written $\pm\sqrt{x}$: either of them may be written $x^{\frac{1}{2}}$, but either of the symbols $x^{\frac{1}{2}}$, $\sqrt{x}$ written alone means the *positive* square root. The symbol $x^{\frac{3}{4}}$ means the cube of the fourth root of x.

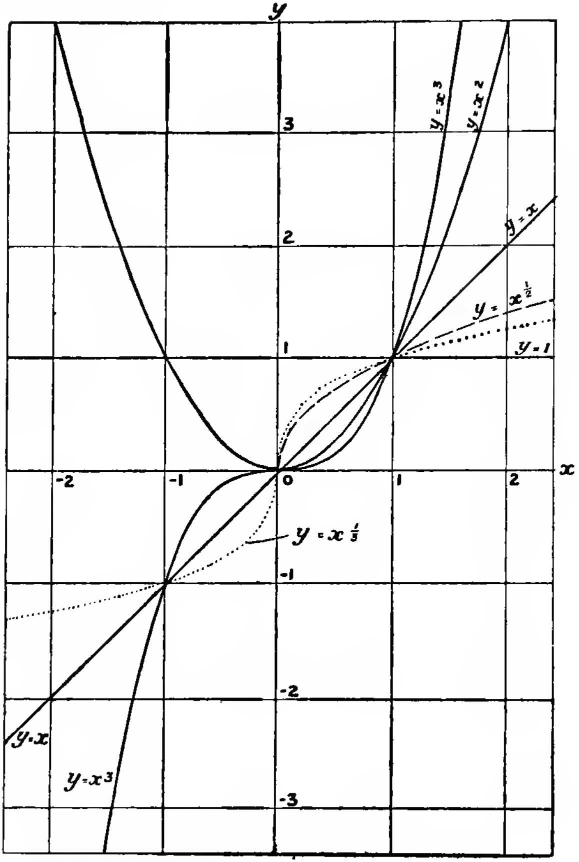

Fig. 24.—Graphs of $y=1$, x, x^2, x^3, $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$.

We shall now discuss a series of typical algebraic laws.

Powers. Positive Integral.—In Fig. 24 are shown the graphs of some *positive integral* powers. The first graph $y = 1$ can be regarded as the *zero power* of x, for $x^\circ = 1$. The second $y = x$ gives the first power, for which the ordinate is always equal to the abscissa. All the other graphs of positive integral powers touch the x-axis at the origin. The higher the power, the slower the curve rises at first, and the quicker it rises afterwards. They all pass through the point $x = 1$, $y = 1$. When x is negative, all the even powers, $y = x^\circ, x^2, x^4, \ldots$ have positive values, and the odd powers $y = x, x^3, x^5, \ldots$ have negative values. The even power graphs are all *symmetrical* about the y-axis, that is to say, the left hand side is obtained by reflecting the right hand side in the y-axis. The negative power graphs are *skew-symmetrical* about the origin, the left hand side being obtained by two reflexions of the right hand side, the first in the y-axis, and the second in the x-axis. This is expressed algebraically,

$$y = (-x)^2 = +x^2 \text{ for an even power}$$
$$\text{and } y = (-x)^3 = -x^3 \text{ for an odd power.}$$

Positive Fractional Powers.—It should first be noticed that for fractional powers in which the denominator is even, the curve has no left hand side: for example if in $y = \sqrt{x} = x^{\frac{1}{2}}$, we put a negative number for x, there is *no value* of y for there is no square root of a negative number.[1] The graph of $y = x^{\frac{1}{2}}$ is seen to stop short at the origin. When the denominator is odd there are values of y for negative values of x. This is seen in the graph of $y = x^{\frac{1}{3}}$.

---

[1] In higher work we get roots of negative numbers, by *inventing* new numbers, called the *imaginary numbers.* The first of these is written i and signifies $\sqrt{-1}$. An imaginary number cannot be shown in an ordinary graph.

The graphs of the powers less than unity are above the straight line $y = x$, between $x = 0$ and $x = 1$, and beyond $x = 1$ they are below $y = x$. The graph of $y = x^{\frac{3}{2}}$ is intermediate between $y = x$ and $y = x^2$; the graph of $y = x^{2\frac{1}{2}}$ is intermediate between $y = x^2$ and $y = x^3$.

The graph of $y = x^{\frac{1}{3}}$ may be written $y^3 = x$, and hence it is the same curve as the graph of $y = x^3$, from which it is obtained by interchanging x and y.

Negative Powers.—For any negative power say $y = \frac{1}{x}$, the ordinate becomes indefinitely great, when the abscissa becomes indefinitely small (positive or negative). The curve does not cut the y-axis at all but gets indefinitely near to it at a great distance from O.

A straight line that a curve approaches indefinitely without ever meeting is called an *asymptote*.

The curves $y = \frac{1}{x}, \frac{1}{x^3}, x^{-\frac{1}{3}}$, etc., approach the y-axis in opposite directions, while $y = \frac{1}{x^2}, \quad x^{-\frac{2}{3}}$, etc., approach it in the same direction.

The curve $y = x^{-\frac{1}{2}}$ lies on one side of the asymptote, and approaches it in one direction only, as $x^{-\frac{1}{2}}$ does not exist for negative values of x.

The x-axis is also asymptotic to these curves.

The curve $y = \frac{1}{x}$ (or $xy = 1$) is especially important: it is a *rectangular hyperbola*. An important property of this curve is that the rectangle contained by the co-ordinates (xy) is constant in area. The origin is the *centre.*

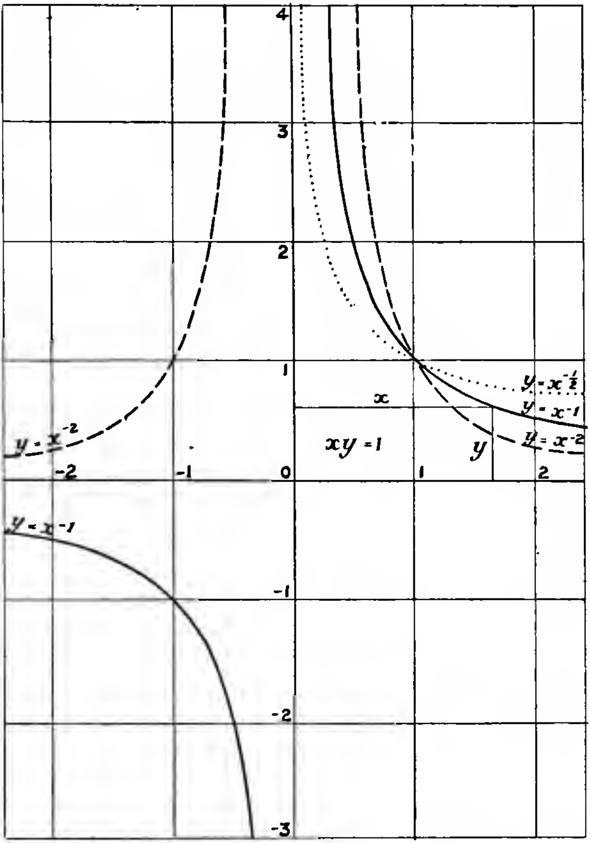

FIG. 25.—Graphs of $y = x^{-\frac{1}{2}}, x^{-1}, x^{-2}$.

Curves of similar character are obtained by changing the ordinate in any ratio. If the ordinate is multiplied by c, then all these curves are included in the general formula

$$y = cx^n$$

where c and n may have any values positive or negative. When c is negative, the curves are on the opposite side of the x-axis.

These curves are of importance in connexion with gas and vapour pressures.

§ 11. **Linear Laws.**—These are the laws that contain no fractions involving the variable, and no powers higher than the first: the general form is

$$y = mx + c$$

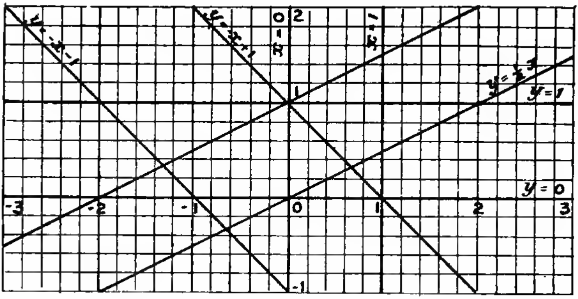

FIG. 26.—Graphs of $y = 0$, $x = 0$, $y=1$, $x = 1$, $y = \frac{1}{2}x$; $y = -x + 1$; $y = -x - 1$.

where m and c have any values whatever positive or negative.

All the graphs are *straight lines* and they represent quantities that are increasing or decreasing at a constant rate. The straight line represented by

$$y = \tfrac{1}{2}x$$

passes through the origin and its slope is $\frac{1}{2}$. The graph of

$$y = \tfrac{1}{2}x + 2$$

is the same straight line as before, but raised through a height of 2 units. In general m is the slope of the straight line, and c is the distance from the origin

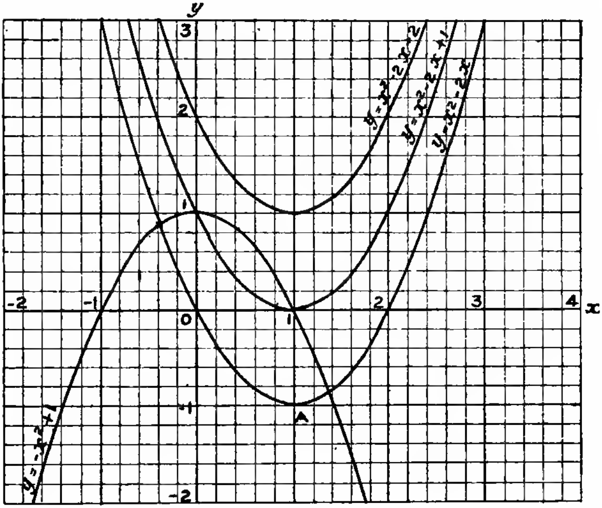

FIG. 27.—Graphs of $y = x^2 - 2x, x^2 - 2x + 1, x^2 - 2x + 2, y = - x^2 + 1$.

where it cuts the y-axis. If m is negative the straight line slopes downwards. If c is negative it cuts the y-axis below the origin. The *equations* of the axes themselves are

$$y = 0 \text{ (the } x\text{-axis) and } x = 0 \text{ (the } y\text{-axis).}$$

It will be noticed that a straight line parallel to the y-axis has to be represented by an equation in which y does not appear, for instance,

$$x = 1 \text{ (in the figure).}$$

§ 12.—**Quadratic Laws.**—These are the laws containing no power of x higher than the second, and no fractions in x; their general form is

$$y = ax^2 + bx + c$$

where a, b, c are any constant numbers, positive or negative. *The graphs are always parabolas with their axes vertical.* The parabola is especially important as it occurs frequently in the work of every engineer.

Taking first the equation

$$y = x^2 - 2x,$$

we note that the graph passes through the origin: this is always the case when no term occurs in the equation independent of both x and y (constant term). When x becomes very large, y becomes large at the same time, for after a time x^2 is incomparably larger than $2x$: hence the curve rises indefinitely to the right. It rises to the left also, for when x is negative x^2 is still positive. It follows that the curve has a minimum, the point A.

If we add a constant (unity) to the right hand side of the equation we obtain $y = x^2 - 2x + 1$, which represents the same curve as before, but raised higher up. The graph of

$$y = x^2 - 2x + 2$$

is again the same curve, two units higher.

When the curve crosses the x-axis, we have $y = 0$, so that on the x-axis for the first curve

$$x^2 - 2x = 0.$$

Hence *the abscissæ of the points where the curve cuts the x-axis are the roots of this equation viz.* $x = 0, 2$. Note that the effect of raising the curve bodily is first to bring the two distinct roots into coincidence, and then to make them disappear.

REVERSED PARABOLAS.—The effect of reversing all the signs on the right hand side is to reverse all the ordinates. A law in which the coefficient of x^2 is negative, for example

$$y = -x^2 + 1,$$

where it is -1, will be represented by a curve having its ends pointing downwards instead of upwards. It will consequently have a maximum instead of minimum.

SLOPE OF PARABOLA.—An important property of these curves is that their slopes are always increasing (decreasing in the last case) at a constant rate. The slope-curves are therefore straight lines. The first three curves will all have the same slope curve ($y = 2x - 2$), for raising or lowering a graph does not affect the slope. In the case of shear and bending moments for beams, for example, to which we have referred already (p. 19), the shear diagram is the slope curve of the B. M. diagram, and the B. M. diagram is the sum curve of the shear diagram. In the case of a uniformly distributed load, the shear diagram is a straight line; therefore the B. M. diagram will be a parabola.

A parabola has no point of inflexion.

§ 13. **Cubic Laws.**—These have the general form

$$y = ax^3 + bx^2 + cx + d.$$

Taking the typical one

$$y = x^3 - x^2 - x + \tfrac{1}{2},$$

it will be noted that for very great values of x, either positive or negative, the term x^3 will dominate the others. Now when x is negative, x^3 is also negative, so that the curve will descend rapidly on the left-hand side, and similarly it will rise rapidly on the right-hand side.

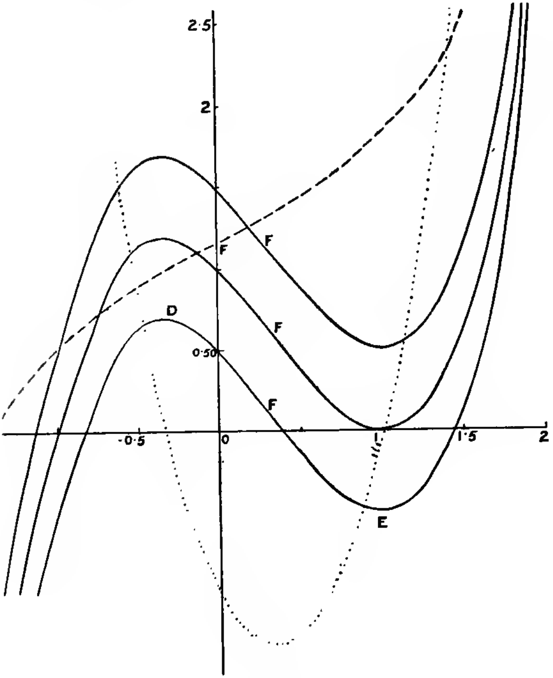

Fig. 28.

$$\left.\begin{aligned}\text{Graphs of } y &= x^3 - x^2 - x + \tfrac{1}{2} \text{ (lower curve)}\\ &= x^3 - x^2 - x + 1\tfrac{1}{2} \text{ (centre curve)}\\ &= x^3 - x^2 - x + 2\tfrac{1}{2} \text{ (upper curve)}\end{aligned}\right\} \text{———}$$

$$y = \frac{x^3}{6} - \frac{x^4}{2} + x + 1 \qquad \text{- - - - -}$$

$$\text{(slope curve) } y = 3x^2 - 2x - 1 \qquad \cdots\cdots$$

This curve crosses the axis three times so that the equation

$$x^3 - x^2 - x + \tfrac{1}{2} = 0$$

has three roots. If the curve is raised, two of these will become coincident and then disappear; two roots (the first two) will also become coincident and disappear if the graph is lowered; but there is always at least one root.

The curve has a maximum D and a minimum E, these being the "critical points". But this is not always the case with cubic curves. The curve

$$y = \frac{x^3}{6} - \frac{x}{2} + x + 1$$

has neither a maximum nor a minimum; it will be remarked that the curve bends over as if to form a maximum, but changing its intention, bends back again (F). A cubic cannot have a maximum without having a minimum also.

Point of Inflexion.—A cubic always has one point of inflexion F. The tangent at this point is inclined downwards when there is a maximum and a minimum, and is horizontal or inclined upwards, when there is no maximum or minimum.

Slope-Curve.—The slope-curve of a cubic is a quadratic, i.e. a parabola: it cuts the x-axis at the values of x corresponding to the maximum and minimum when these points exist.

If the coefficient of x^3 is negative, the curve ascends on the left and descends on the right.

Quartic and Higher Laws.—These are of a similar nature to the cubic and quadratic. We will not discuss them in detail, but remark that an $n-ic$, that is to say a curve whose equation is of the form

$$y = ax^n + bx^{n-1} + cx^{n-2} + \dots$$

cuts the axis at n points at most: it has at most

$n - 1$ critical points, and at most $n - 2$ points of inflexion.

§ 14. **Some other Algebraic Laws.**—The algebraic laws which occur are very varied and it would be impossible to classify them completely. We will content ourselves with giving a few more by way of illustration. When it is required to draw the graph of a given law, it always has to be done ultimately by *finding points* on the curve. But in each case the general shape may be determined, and its drawing very much facilitated by noting its *special features* such as :—

1. The points where it crosses the axes.
2. Its critical points.
3. Its points of inflexion.
4. Its asymptotes.
5. The parts which rise and those which fall.
6. Its behaviour for very large positive and negative values of x.
7. Its symmetry about the x-axis, or the y-axis.

These features have all been mentioned already, and some will be discussed in more detail later on.

We will remark here that a curve is symmetrical about the y-axis when only even powers of x occur in its equation, for instance, $y = x^2$, $y = \frac{1}{x^2 - 1}$.

The graph of $y = \frac{1}{x - 1}$ is the same curve as the graph of $y = \frac{1}{x}$ already given in Fig. 25, but it is moved from left to right parallel to the x-axis through a distance of one unit.

The graph of $y = \frac{1}{x^2 - 1}$ has three asymptotes. The first is the x-axis which the curve approaches indefinitely on the upper side both to the left and to the right. The curve rises indefinitely as we approach $x = -1$ and $x = +1$. These vertical lines are therefore also asymptotes. The graph has one critical point, a maximum $y = -1$ occurring when $x = 0$.

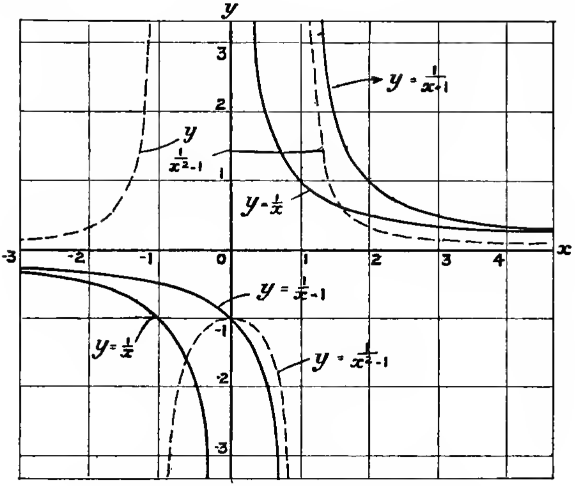

FIG. 29.—Graphs of $y = \frac{1}{x}$, $\frac{1}{x-1}$, $\frac{1}{x^2-1}$

§ 15. **Implicit Laws.**—Very often the dependent variable is given not directly in terms of x (explicitly) as in all the cases considered up to now, but implicitly, that is to say, a general relation is given between x and y. For example we may have

$$\frac{x^2}{4} + y^2 = 1 \quad . \; . \; . \; (1)$$

An implicit relation may often be converted into an explicit relation by regarding it as *an equation for determining* y, and solving. For instance the relation

$$xy - 1 = 0$$

when solved becomes

$$y = \frac{1}{x}.$$

If we take

$$\frac{x^2}{4} + y^2 = 1,$$

this may be written

$$y^2 = 1 - \frac{x^2}{4}$$

so that

$$y = +\sqrt{1 - \frac{x^2}{4}} \text{ or } y = -\sqrt{1 - \frac{x^2}{4}} \quad . \quad . \quad . \quad (2)$$

The implicit relation in this case has to be replaced by *two* explicit relations. It is generally more convenient and often necessary to deal with an implicit law as it stands and not to convert it into one or more explicit laws.

The curve (1) is an ellipse (Fig. 30); both axes are axes of symmetry. The *lengths of the semi-axes* are 2 and 1 units respectively. By inspecting (2) it will be seen that the curve cannot exist when $x > 2$. The two equations of (2) give the upper and lower halves of the curve respectively.

The general equation of the ellipse whose semi-axes are a and b is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The curve

$$y^2 = x(x - 1)(x - 2)$$

cannot exist when x is less than zero, or when x lies between 1 and 2, for such values of x would make y^2 a negative quantity. The curve consists of an oval and an infinite branch. (Although there is an infinite branch, there are no asymptotes.)

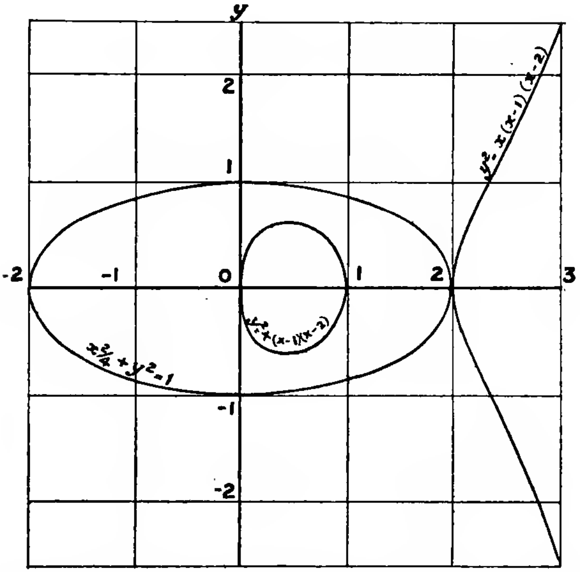

Fig. 30.—Graphs of $\frac{x}{4}+y^2=1$ and $y=x\ (x-1)\ (x-2)$. [Note that the latter has two branches.]

II. Trigonometric Formulæ.

§ 16.—**The Sine Law.** Definite laws connecting two variables can be obtained by other means than by algebraical relations: they can for example be defined geometrically. Trigonometric laws are of this kind.

Expressing the angle $A\hat{O}P = x$ in circular measure, so that

$$x = \text{arc } AP/OP$$

and writing

$$y = \sin x = NP/OP,$$

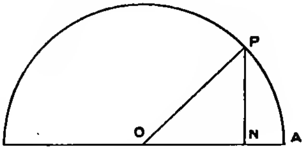

Fig. 31.

we have a law connecting the value of y with that of x.[1]

This is a new law, a perfectly definite one, which cannot be expressed algebraically. (See Figs. 15 and 32.)

A very similar law is

$$y = \cos x = ON/OP.$$

The graph of this is in fact not distinct from the sine curve, for remembering that

$$\cos x = \sin \left(x + \frac{\pi}{2}\right),$$

we see that $y = \cos x$ is simply the curve $y = \sin x$,

---

[1] We wish to emphasize the necessity for taking the correct scale for the sine curve. The abscissæ are the angles in circular measure, the ordinates are the actual values of the sine. The height of the sine curve is unity, and the length of the first arch is $\pi = 3{\cdot}142$. Although other scales are sometimes convenient in practice, this is the only one giving the true sine law, and the discussion following applies only to this curve.

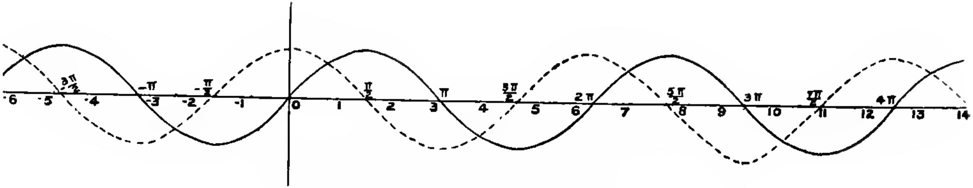

FIG. 32.—Sine Curve (full), Cosine Curve (dotted).

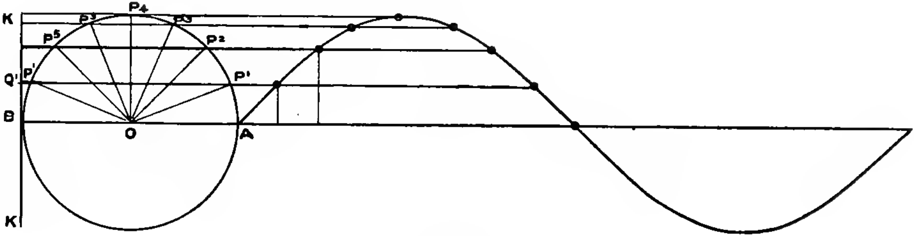

FIG. 33.

moved back horizontally through a distance of $\frac{\pi}{2}$ units. (See Figs. 14 and 32.)

§ 17.—**Harmonic Motion.**—This is a good illustration of the sine law.

Suppose a point P to move at a uniform rate (of a radian per second, say) round the circumference of a circle (Fig. 33). Let it start from the point A and let Q be its projection on a vertical KK. At first Q will move at the same rate as P, but its speed will diminish as it rises until it comes to rest at K and then descends. To obtain the displacement curve for this motion, we have only to measure out to the right on the horizontal line through A, the arcs through which P moves: since the speed of P is unity, the abscissæ are the times. We then erect ordinates equal to the ordinates of P. The curve obtained is clearly the sine curve. We shall return to harmonic motion later (see p. 74).

§ 18.—**Properties of the Sine (or Cosine) Law.**—The first property is its *periodicity*. The part of the sine curve which lies between $\theta = o$ and $\theta = 2\pi$ is repeated between $\theta = 2\pi$ and $\theta = 4\pi$ and again repeated between $\theta = 4\pi$ and $\theta = 6\pi$, and so on (Fig. 32). *The sine curve is periodic, its period being* 2π. This is expressed by the formula

$$\sin(\theta + 2\pi) = \sin\theta.$$

Similarly, of course,

$$\cos(\theta + 2\pi) = \cos\theta.$$

The second property is its symmetry. An inspection of the graph will show that the left-hand side of the sine curve is obtained from the right by reflexion first in the y-axis, then in the θ-axis. So that *the sine* *curve is skew-symmetrical about the origin* (cf. Fig. 24, § 10). The sine-curve is also skew-symmetrical at the points $\theta = \pi, 2\pi, \ldots -\pi, -2\pi, \ldots$: it is *symmetrical* about the verticals $\theta = \frac{\pi}{2}, \theta = \frac{3\pi}{2}, \theta = -\frac{\pi}{2}$, etc.

The cosine curve similarly is symmetrical about the y-axis, and about the verticals $\theta = \pi, 2\pi \ldots$ while it is skew-symmetrical at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

These facts are expressed by the formulæ

$$\sin(-\theta) = -\sin\theta;\ \cos(-\theta) = \cos\theta, \text{ etc.}$$

which may at once be deduced geometrically from a figure similar to Fig. 31.

§ 19.—Theorem in Limits.—A third very important property which may be seen in the graph (Fig. 34) is that *the sine curve cuts the axis at an*

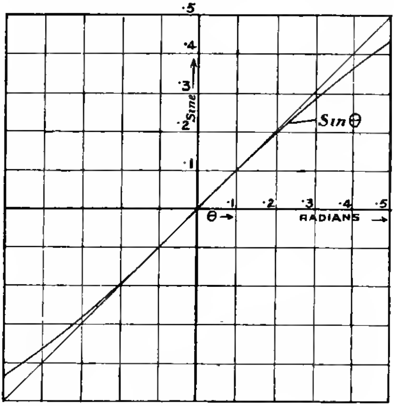

Fig. 34.

angle of 45° $\left(=\frac{\pi}{4}\right)$. This means that near the origin the sine curve

$$y = \sin \theta$$

coalesces with the straight line

$$y = \theta,$$

so that when θ is small the ratio

$$\frac{\sin \theta}{\theta} = \frac{\text{ordinate of sine curve}}{\text{ordinate of straight line } y = \theta} \to 1$$

approximates to unity. It is very important to remember that θ is in circular measure.

A proof of this important property will be found in textbooks on trigonometry.

§ 20.—**The Tangent.**—We define the tangent of x as follows:

$$\tan x = \frac{\sin x}{\cos x}.$$

The graph is shown in Fig. 35. Whenever $\cos x$ becomes indefinitely small, $\sin x$ approaches unity, positive or negative, and $\tan x$ therefore becomes indefinitely great. Hence the graph has vertical asymptotes at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

When $\sin x$ approaches zero, $\cos x$ approaches unity. Hence the graph crosses the x-axis at the same points as the sine curve does, and at the same angle, namely 45°. The ratios

$$\frac{\tan x}{\sin x} = \frac{1}{\cos x}$$

and

$$\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x}$$

both approach unity when x is small.

The Cosecant is defined by

$$\operatorname{cosec} x = \frac{1}{\sin x},$$

the secant by

$$\sec x = \frac{1}{\cos x},$$

and *the cotangent* by

$$\cot x = \frac{1}{\tan x}.$$

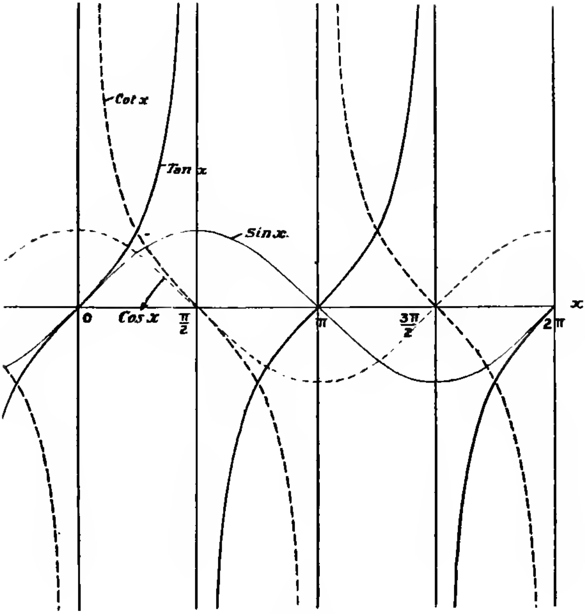

Fig. 35.

The nature of these curves can be inferred easily from the graphs already discussed. They all have vertical asymptotes. The curve cosec x has maxima where sin x has minima and minima where sin x has maxima.

§ 21. **Inverse Relations.**—We can of course regard sin x as independent and x as depending upon it; given any suitable value of $\sin x = y$ we can find the corresponding value of x. This relation is written

$$x = \sin^{-1} y.$$

$$\begin{aligned} \text{For } y &= 0 & x &= 0^\circ = 0 \text{ radians} \\ y &= \frac{1}{\sqrt{2}} & x &= 45^\circ = \frac{\pi}{4} \text{ radians} \\ y &= 1 & x &= 90^\circ = \frac{\pi}{2} \text{ radians,} \end{aligned}$$

and so forth.

The notation $\cos^{-1} y$, $\tan^{-1} y$, etc., is used similarly.

Note that although $\sin^2 x$ means $(\sin x)^2$, yet $\sin^{-1} x$ does not mean $(\sin x)^{-1} = 1/\sin x$. This confusing notation is unfortunate, but is firmly established. On the Continent $\sin^{-1} x$ is written *arcsinx.*

Exercises 2.

1. Draw the graph of the function

$$y = 2x^3 - 9x^2 - 12x + 24.$$

Find for what values of x this function has maximum and minimum values and where its point of inflexion occurs.

2. Rankine's formula for the safe working stress in tons per sq. in. upon mild steel columns is of the form $f_P = \dfrac{6}{1 + \dfrac{c^2}{6000}}$, where c is the "buckling factor".

Plot values of f_P against c and determine the point of inflexion.

3. The bending moment at a distance x from one

end of a beam of span l loaded with a uniformly increasing load W is equal to $M = \frac{Wx}{3} - \frac{Wx^3}{3l^2}$.

Plot the bending moment diagram for this beam, finding the maximum bending moment and the point at which it occurs.

4. The horse-power transmitted by a certain wire rope is given by the formula $\text{H.P.} = \frac{(62790 - 3v^2)\,v}{230230}$ where v is the velocity in feet per second.

Plot H.P. against v and find the velocity for which it is a maximum.

5. Plot the formula $y = x^2 - 4{\cdot}2\,x + 2{\cdot}93$ and find from your plotting the solution to the equation $x^2 - 4{\cdot}2\,x + 2{\cdot}93 = 0$.

6. Find, by plotting, the value of x which will make $3 \sin x + 2 \cos x$ a maximum.

7. Plot the equation $y = x\,(81 - x^2)$ and find the maximum value of y and the value of x at which it occurs.

8. The area of the segment of a circle whose arc is l in length and subtends an angle θ at the centre is equal to $\frac{1}{2}\,l^2\left(\frac{1}{\theta} - \frac{\sin\theta}{\theta^2}\right)$, θ being in radians.

Plot the area against θ and find where it is a maximum.

9. Plot $y = \sin x\,(1 + \cos x)$ and find where it is a maximum.

10. The cost in pounds for running a steamer a certain journey is $C = \frac{1300}{v} + v^2$ where v is the speed in knots.

Plot the cost against the speed and find where it is a minimum.

ANSWERS TO EXERCISES.

2.

1. $x = \cdot 56$ and $3 \cdot 56$. Point of inflexion at $x = 1 \cdot 5$. **2.** $c = 44 \cdot 7$. **3.** Max. B.M. $= \cdot 128 Wl$ at distance $\cdot 577 l$ from one end. **4.** 83·5 feet per sec. **5.** 3·32, ·88. **6.** 56° 18′. **7.** 280·6; $x = 5 \cdot 2$. **8.** $\theta = \pi$. **9.** $x = \frac{\pi}{3}$. **10.** 8·7 knots.

CHAPTER III.

LAWS EXPRESSED SYMBOLICALLY (CONTINUED).

III. EXPONENTIAL AND LOGARITHMIC LAWS.

§ 22. **The Compound Interest Law.**—Suppose a sum of money (£y_0, say) were put out at 5 per cent. compound interest. At the end of each year would be added to the sum one-twentieth of its amount at the beginning of that year. The growth of the amount could be represented by a graph which would rise by a small jump or step at the end of each year (Fig. 36).

In the diagram distances measured along the horizontal axis represent times in years; the ordinate represents the amount and remains constant during each year, but increases suddenly at the end. The *step* increases in height from year to year, *each step being proportional to the corresponding ordinate* (the figure is illustrative only; the actual graph is much flatter).

If now the interest were added continuously instead of in instalments at the ends of years, at a rate always proportional to the ordinate, and such that the total interest per annum were the same as before, we should have a smooth curve drawn through the upper corners of the steps. This curve represents the *compound interest law*, and its characteristic property is that its rate of increase or slope is proportional to its ordinate. This property may be expressed by the formula

$$\frac{dy}{dx} = ky \quad . \quad . \quad (1)$$

and *all quantities such that their rates of increase are proportional to themselves follow such a law.*

We can easily find the amount at the end of each year. If the amount is initially £y_o, then at the end of the 1^{st}, 2^{nd}, 3^{rd}, . . . nth . . . year the amount will be $y_o.1{\cdot}05$, $y_o(1{\cdot}05)^2$, $y_o(1{\cdot}05)^3$, . . . $y_o\,(1{\cdot}05)^n$. . . .

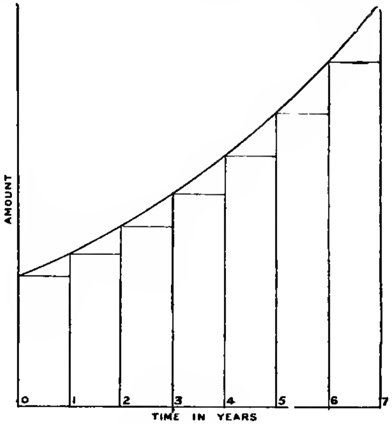

FIG. 36.

Now the powers to which 1·05 is raised, namely 1, 2, 3, . . . n, . . . are simply the *abscissæ*—if one unit is made to represent one year. Putting in the intermediate values, i.e. *interpolating*, we see that the ordinate of the smooth curve is

$$y = y_o\,(1{\cdot}05)^x$$

where x is *any* abscissa (and not necessarily a whole number).

The general law, when the rate of interest is anything we please, is

$$y = y_0 a^x, \quad . \quad . \quad . \quad (2)$$

on putting a for 1·05. This law has the characteristic property (1), but we shall not at present find the connexion between the number a and the ratio of proportionality k. Since the law is represented by a variable power of a number a, it is also called *exponential*, x being the *exponent* of the power.

It might be thought at first sight that this law was an algebraic one, but this is not so, for in this case the power x is variable. In algebraic expressions, such as x^n, the power is constant.

§ 23. **Examples** of quantities which follow the exponential laws will be familiar to the student. We take some specific cases.

1°. Rate of Growth of a Population.—If the rate of increase per thousand is constant, the whole population will increase at a rate proportional to itself at any time.

2°. Friction Between Belt and Pulley.—Suppose a belt ARQB to pass round a circular pulley O, and to be about to slip in the direction named. Let

T_0 = tension in belt at A
T = " " " " R
$T + \delta T$ = " " " " Q
P = pressure per unit length at R.
r = radius of pulley.
θ = angle $A\hat{O}R$.
$\delta\theta$ = angle $R\hat{O}Q$.
μ = cœfficient of friction.

The forces acting upon the element RQ of the belt are

$Pr\delta\theta$ direction of OR (approximately).
T ,, tangent at R, backwards.
$T + \delta T$,, tangent at Q, forwards.
$\mu Pr\delta\theta$,, tangent at R, backwards (approximately), the frictional force.

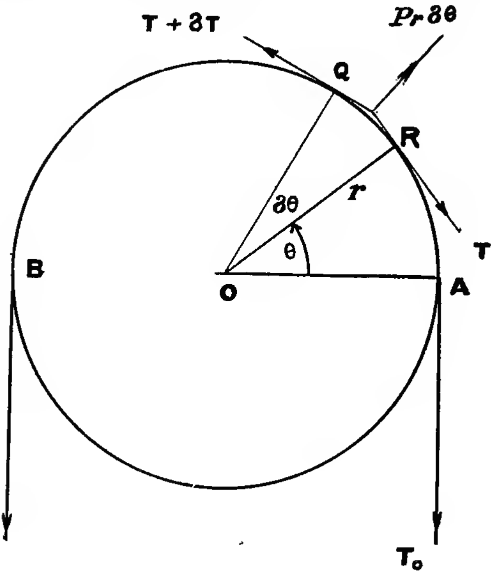

FIG. 37.

Resolving in the direction OR

$$Pr\delta\theta = (T + \delta T)\sin\delta\theta.$$

We may put $\delta\theta$ for $\sin\delta\theta$, and then neglect the product $\delta T \,.\, \delta\theta$; thus

$$Pr\delta\theta = T\delta\theta \text{ or } T = Pr \quad . \quad . \quad . \quad (1)$$

Resolving along the tangent at R

$$T + \mu P r \delta\theta = (T + \delta T) \cos \delta\theta.$$

Put 1 for cos $\delta\theta$.

$$T + \mu P r \delta\theta = T + \delta T,$$

or $$\delta T = \mu P r \delta\theta$$

i.e. $$\frac{\delta T}{\delta\theta} = \mu P r = \mu T \text{ by (1).}$$

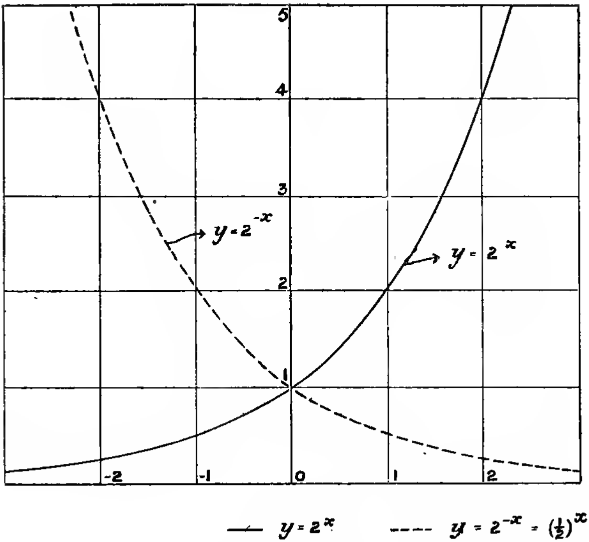

FIG. 38.—$y = a^x$ $(a < 1)$ *or* $y = b^{-x}(b > 1)$
(y_o is taken = 1 in this figure).

Thus the rate of increase of the tension with the angle is proportional to the tension, which must therefore increase round the pulley exponentially

$$T = T_o a^\theta *$$

for a suitable value of a.

We shall mention some other cases later.

---

* The formula is precisely $T = T_o e^{\mu\theta}$ (see below).

§ 24. **Decreasing Quantities.**—A quantity also follows an exponential law when its *rate of decrease* is proportional to its value at any time. In fact a rate of decrease is only a *negative* rate of increase. But in this case the number a will be less than unity. We have thus for such a quantity

$$y = y_o\, a^x \qquad a < 1$$

and

$$\frac{dy}{dx}\,(\text{negative}) = -\,ky$$

where k is some positive number.

Another way of looking at this case is to put $a = \frac{1}{b}$, so that $b > 1$. Then

$$y = y_o a^x = y_o \left(\frac{1}{b}\right)^x = y_o \frac{1}{b^x} = y^o b^{-x}.$$

From this point of view we see that the curve is exactly the same as the rising curve $y = y_o b^x$, but reflected in the vertical axis (because opposite signs are taken for x).

§ 25. **Further Examples.**—3°. *Fall of electric current in a circuit from which the E.M.F. is suddenly cut off.* When an electric current C is flowing round a closed circuit, a magnetic field is set up, the number N of lines of magnetic force linking the circuit being proportional to the current. Again if the number N of lines of magnetic force is diminished, then a current is set up in the circuit, its direction being the same as that of the current mentioned in the last sentence, and its magnitude being proportional to the rate of diminution of the number of lines of magnetic force. Therefore in the case in view, the current C at any instant is proportional to the rate of diminution of N, i.e. to the rate of diminution of C. So

$$C = -\lambda \frac{dC}{dt} \quad (t = \text{time in seconds})$$

and the current follows a diminishing exponential law.

4°. *Damped Vibrations.*—These will be discussed shortly (p. 75). We mention here that the amplitude A of each vibration is a given fraction, say a (< 1), of the preceding one. Thus successive amplitudes will be

$$A_o,\ A_oa,\ A_oa^2,\ A_oa^3,\ \ldots$$

and in general

$$A = A_oa^x \qquad a < 1$$

(x being here a whole number, viz. the number of vibrations) so that the amplitudes diminish exponentially.

§ 26. **Special Exponential Law.**—We have seen that when y follows the law

$$y = y_oa^x,$$

then its rate of increase is proportional to itself

$$\frac{dy}{dx} = ky \quad . \quad . \quad . \quad (1)$$

It is clear that there will be one special value of a for which $\frac{dy}{dx}$ is not merely proportional to y, but actually equal to it,

$$\frac{dy}{dx} = y.$$

This value of a is of the utmost importance; it is written e and

$$a = e = 2{\cdot}718281\ldots$$

The number e is not exact or *commensurable,* that is to say it cannot be represented exactly by an ending decimal or by a fraction, but only approximately: we have given it to one part in a million.

Taking *any value* of a we could write

$$a = e^{\lambda}$$

where λ is the power to which we must raise e in order to get a (or λ is the logarithm of a to the base e—see next paragraph). Of course λ also will not be a commensurable number in general. Suppose, for instance, a had the value 10, then

$$\lambda = 2{\cdot}302585\ \ldots$$

so that

$$10 = (2{\cdot}718281\ \ldots)^{2{\cdot}302585\ \ldots}$$

to any required degree of approximation. The calculation by which λ is determined will be easy to any one familiar with logarithms, and will be briefly explained below.

Thus we may always write

$$y = y_o\,e^{\lambda x} \quad . \quad . \quad . \qquad (3)$$

instead of (2).

It will now be clear that λ is precisely the number k. For from the above the rate of increase of y *with respect to* λx is y: so the rate of increase of y with respect to x is λy. So that

$$k = \lambda \qquad \frac{dy}{dx} = \lambda y = \lambda y_o\,e^{\lambda x} \quad . \quad . \quad . \qquad (4)$$

(If the student does not see this at once he should repeat it putting a simple value, say 2, for λ.)

Finally then the two equations (2) (p. 65) and (3) are equivalent to each other.

Thus our final formula for the tension in the belt in the example considered above (2°, p. 65) is $T = T_o e^{\mu\theta}$.

When a is less than 1, *the value of* λ *will be negative.* In the example (3°) above the current is

$$C = C_0 e^{-\lambda t}.$$

§ 27. **Logarithmic Laws.**—If we consider powers of a number, say 10,

$$y = 10^x,$$

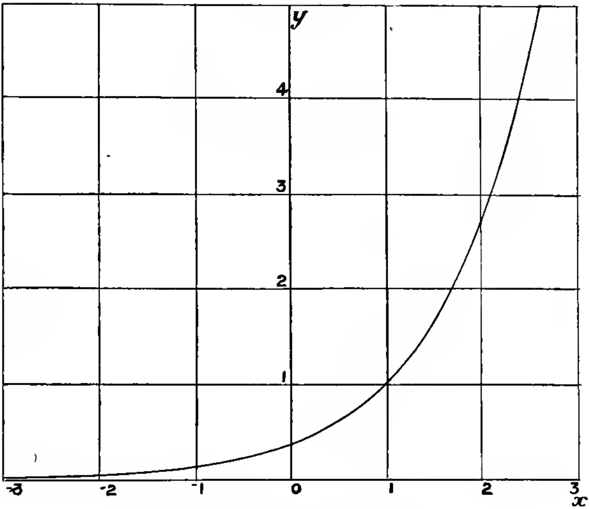

Fig. 39.—$y = e^x$ ($- y_0 = 1$).

we can, as we have seen, plot them to a curve (similar to Fig. 38) * and we make the following observations:—

The curve is always rising.

When $x = 1$, $y = 10$.

When $x = 0$, $y = 1$.

---

* This curve is given below (Fig. 40), differently arranged for reasons explained. The student will follow the first part of the reasoning better from Fig. 38, or better still from a figure of his own for $y = 10^x$.

When x increases from 0 to 1, y increases from 1 to 10.

When x is greater than 1 ($x > 1$), y is greater than 10, and y increases indefinitely with x.

When x is negative ($x = -z$), then $y = 10^{-z} = 1/10^z < 1$. The ordinate y becomes indefinitely small, but never vanishes or becomes negative, when x becomes an indefinitely great negative number.

Thus y takes every positive value for some value of x.

Conversely given any value of y, we can always find a power of 10 equal to this value of y; this is called *the logarithm of y to the base* 10, and is written shortly

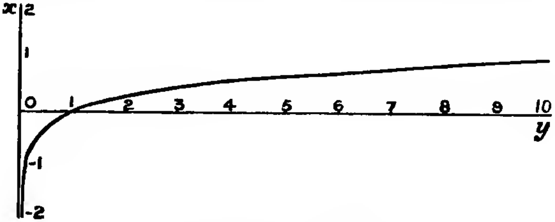

FIG. 40.—Graph of $x = \log_{10} y$.

$$\log_{10} y.$$

So that

$$y = 10^x \text{ and } x = \log_{10} y$$

are two ways of writing the same thing.

The graph of $y = 10^x$, will give at once the logarithm of any number: thus to find $\log_{10} 2$, find the ordinate equal to 2 in the figure, and the abscissa will be $\log_{10} 2$. Or we may draw the graph of the logarithm

$$x = \log_{10} y$$

of course making the y (abscissa) axis horizontal, and the x-axis vertical (Fig. 40). But the graph is really the same as before.

Logarithms to base 10 are *common logarithms;* they are used in arithmetical calculations. Logarithms to the base $e = 2{\cdot}71828\ldots$ are also very important: they are *Napierian or Hyperbolic logarithms* and are almost invariably used in theoretical work, for which they are more convenient.

To convert from common logarithms to Napierian logarithms we use the rule

$$\log_e x = 2{\cdot}303 \log_{10} x.$$

We have remarked that logarithmic laws and exponential laws are not really distinct: they are *inverse to each other,* just as $y = \sin x$ and $x = \sin^{-1} y$ are inverse to each other. When, as is usual, a law can be expressed in either an exponential or a logarithmic form, the former is preferred. The symbol $\log x$ implies $\log_e x$ in theoretical work and $\log_{10} x$ in numerical work.

The following operation should be noted

$$e^{\log_e x} = x\,;\ \log_e e^x = x.$$

IV. MIXED LAWS.

§ 28. The Preceding Laws.—

I. $1, x, x^n, \dfrac{1}{1+x}$, etc. . . .

II. Sin x, cos x, tan x, etc.

Sin$^{-1} x$, cos$^{-1} x$, tan$^{-1} x$, etc.

III. e^x, a^x, $\log_e x$

may be combined in a great variety of ways. We will give some important instances.

ADDITION AND SUBTRACTION.—

$$y = x + \sin \tfrac{1}{2} x.$$

The graph would be obtained by adding the ordinates as in Fig 41.

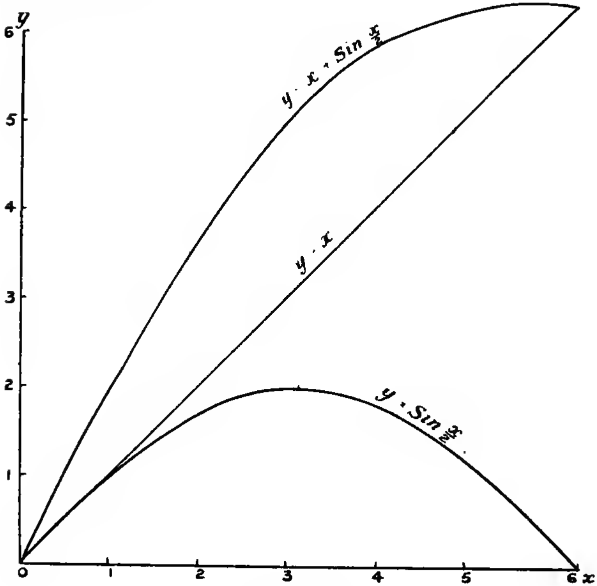

FIG. 41.

MULTIPLICATION AND DIVISION.—Instances are

$$x \sin x\,;\ \frac{1}{x} e^x\,;\ e^x \log_e x;\ \frac{1+x}{\sin x}.$$

The graphs are obtained as before by multiplying or dividing the corresponding ordinates. The following is an example.

DAMPED VIBRATIONS.—This is a very important case; its formula is

$$y = y_o\, e^{-\lambda t} \sin \omega t\,;$$

it is the law followed by all simple * small vibrations whatever, whether mechanical or electrical (see § 25, 4°). The coefficient of sin ωx, viz. $y_0 e^{-\lambda t}$ is the amplitude; it is seen to diminish exponentially.

"FUNCTION OF A FUNCTION".—This is a more subtle combination of laws, and one which it is important thoroughly to understand. As we have noted already instead of saying "y follows such and such a law with respect to x," we may say "y is a function of x": thus x^n, sin x or e^x are all *functions of* x. The general definition is as follows:—

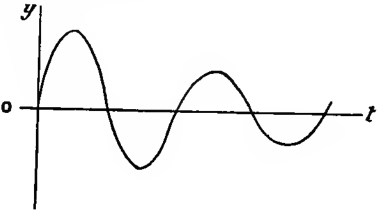

FIG. 42.—Damped vibrations.

A quantity y *is a (one-valued) function of* x, *when to every value of* x *corresponds one value of* y. The statement: "y is a function of x" may be written briefly

$$y = f(x).$$

This might mean $y = e^x$, or $y = \sin x$ or that y was given in terms of x empirically by means of a graph, or a table of values.

If y is a function of x (or depends on x), then x is of course a function of y,

$$x = F(y)$$

---

[*] Strictly "with one degree of freedom". The motion of vibratory systems with several degrees of freedom consists in a combination of several such simple damped vibrations.

since for each value of y there is one (sometimes more than one) value of x. This is the *inverse function;* we use a different letter F, because the law involved is different: this is sometimes written

$$x = f^{-1}(y).$$

If $f(x)$ were e^x, then $f^{-1}(x)$ would be $\log_e x$. We see now the reason for the notation $\sin^{-1} x$, $\cos^{-1} x$, etc. Now consider

$$y = \sin x^n.$$

Suppose we give a value to x, then x^n takes a definite value, and y is the sine of this last value. To express y in terms of x graphically, we first draw the graph of x^n (in the figure $y = x^{\frac{1}{2}}$), then for the ordinate y corresponding to x, take, not the sine of the number x, but the sine of $x^{\frac{1}{2}}$, and so on. This is a *function of a function,* for x^n is a function of x, and y is a function of x^n.

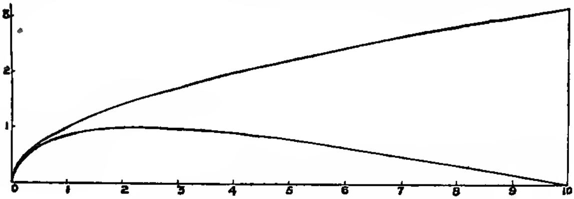

FIG. 43.—$y = x^{\frac{1}{2}}$ and $y = \sin x^{\frac{1}{2}}$.

The law

$$y = (\sin x)^n = \sin^n x$$

is of course different. We first take the sine of x, then we raise it to the nth power: $\sin x$ is a function of x, and y is a function of $\sin x$.

The process may be repeated several times, and may be combined with the processes already mentioned; the expressions

$$\tan^{-1}(e^x \tan e^x), \quad \sin\left(\frac{1}{1 + \log x}\right)$$

explain themselves simply enough, though their graphs are a little difficult to draw.

Another way of looking at the matter is to introduce some more letters. Thus in the function

$$y = \sin x^n$$

we can put

$$z = x^n; \; y = \sin z;$$

then y is a function of z which is itself a function of x.

An example of a rather complicated function involving this idea which occurs in engineering is Hutton's wind pressure formula $P_\theta = P_v \sin\theta^{1\cdot84\cos\theta - 1}$ where P_θ is the wind pressure upon a surface inclined at an angle θ to the horizontal and P_v is the pressure upon a vertical surface.

Exercises 3.

1. Draw a graph of $\log_{10} \sin x$ from $x = 0$ to $x = 90°$.

2. If $x = \frac{1}{2}(e^y + e^{-y})$, show that

$$y = \log_e \{x + \sqrt{x^2 + 1}\}.$$

3. Plot the graph up to $x = 2$ of the function

$$y = \tfrac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{2.3} + \frac{x^5}{2.3.4.5} + \cdots$$

4. Prove that $\sqrt{e} = 1\cdot6487$.

5. Prove that if $x = \log_e \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$

$$\tan\phi = \tfrac{1}{2}(e^x - e^{-x}).$$

6. Plot against a time base the damped vibration given by $y = 2e^{-\cdot3t}\sin 2\cdot98t$. (Values of y should be calculated by logarithms.)

7. Plot against a time base the function

$$y = 10t\,.\,e^{-3t}.$$

ANSWERS TO EXERCISES.

3.

2. $2x = e^y + \frac{1}{e^y}$ $\therefore e^{2y} - 2xe^y + 1 = 0$. Solving this as a quadratic for e^y, $e^y = x \pm \sqrt{x^2+1}$; taking the positive sign (since there is no logarithm of a negative number) $\therefore y = \log_e \{x + \sqrt{x^2+1}\}$.

5. If $x = \log_e \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$, $e^x = \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)$

$$= \frac{1 + \tan\frac{\phi}{2}}{1 - \tan\frac{\phi}{2}} = \frac{\cos\frac{\phi}{2} + \sin\frac{\phi}{2}}{\cos\frac{\phi}{2} - \sin\frac{\phi}{2}} = \text{(multiplying}$$

top and bottom by $\cos\frac{\phi}{2} + \sin\frac{\phi}{2}$)

$$\frac{1 + 2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{\cos^2\frac{\phi}{2} - \sin^2\frac{\phi}{2}} = \frac{1 + \sin\phi}{\cos\phi}$$

Similarly $e^{-x} = \frac{1 - \sin\phi}{\cos\phi}$

$$\therefore e^x - e^{-x} = \frac{1 + \sin\phi}{\cos\phi} - \frac{1 - \sin\phi}{\cos\phi}$$

$$= \frac{2\sin\phi}{\cos\phi} = 2\tan\phi.$$

CHAPTER IV.

SYSTEMATIC DIFFERENTIATION.

§ 29. **Slopes obtained from Formulæ.**—We have seen in the previous chapters that a relation between two varying quantities, or a *law* can be expressed (1) by means of a graph, (2) by means of a formula. The latter method is not only more terse and convenient from many points of view, but it generally throws a great deal of light upon the connexion between two quantities; and this is especially the case when the law is a simple one. In Chapters II and III we summarized the types of formulæ which are most commonly met with. When we have a quantity expressed as a graph, we can find its rate of change graphically, although this is a tedious and rather inaccurate process. We now want to see how when we are given the quantity as a formula we can find the rate of change or differential coefficient in a second formula.

First we will take some simple examples.

1°. $$y = x^2.$$

To obtain the slope $\frac{dy}{dx}$, let us find the change in y when x changes by δx:

for x we have $y = x^2$

for $x + \delta x$ we have $y + \delta y = (x + \delta x)^2$.

The change in $y = \delta y =$ *new value of* y minus *old value of* y

$$\begin{aligned} &= (x + \delta x)^2 - x^2 \\ &= x^2 + 2\,x\delta x + \delta x^2 - x^2 \\ &= 2x\delta x + \delta x^2 \end{aligned}$$

$\therefore$ rate of change of y

$$\begin{aligned} &= \text{change in } y \div \text{change in } x \\ &= \delta y/\delta x = [2x\delta x + \delta x^2]/\delta x \\ &\qquad = 2x + \delta x. \end{aligned}$$

When δx is made indefinitely small the second term in the last expression becomes indefinitely small, and passes the limits of accuracy, so that it can be neglected.[1] Hence *in the limit*

$$\frac{dy}{dx} = 2x.$$

We have thus found a formula for the slope.

2°. $$y = \frac{1}{1 + x}.$$

Here $\delta y = \dfrac{1}{1 + x + \delta x} - \dfrac{1}{1 + x}.$

(bringing to a common denominator)

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[1] Some students find difficulty in following this argument with reference to terms that become so small that they may be neglected. The point to remember is that δx may be as small as we like. Suppose, for instance $x = 10$. If we take $\delta x = \frac{1}{1000}$, $\frac{\delta y}{\delta x} = 20 + \frac{1}{1000}$. If it does not satisfy us to neglect this $\frac{1}{1000}$, take $\delta x = \frac{1}{1{,}000{,}000}$; then $\frac{\delta y}{\delta x} = 20 + \frac{1}{1{,}000{,}000}$. Thus, getting smaller and smaller values of δx, we have finally $\frac{dy}{dx} = 20$, i.e. $\frac{dx^2}{dx} = 2x$.

$$= \frac{(1 + x) - (1 + x + \delta x)}{(1 + x + \delta x)(1 + x)}$$

$$= \frac{1 + x - 1 - x - \delta x}{(1 + x + \delta x)(1 + x)}$$

$$= \frac{-\delta x}{(1 + x + \delta x)(1 + x)}$$

$$\therefore \frac{\delta y}{\delta x} = \frac{-1}{(1 + x + \delta x)(1 + x)},$$

and making δx indefinitely small,

$$\frac{dy}{dx} = \frac{-1}{(1 + x)(1 + x)} = \frac{-1}{(1 + x)^2}.$$

We might differentiate all expressions in this way "from first principles"; it is, however, possible to reduce differentiation to a set of simple rules, and this we shall now do. The method consists in committing to memory a certain number of *standard slopes* (obtained in §§ 31, 32) and deducing all other slopes from these by means of the formulæ in the next article.

§ 30. **General Formulæ.**—We now show how to obtain the slope for the sum (or difference), the product or the quotient, of two quantities, when we know the slopes of these quantities separately.

1°. SUM: if $y = y_1 + y_2$,

$$\text{then } \frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}.$$

For we have $\delta y = \delta y_1 + \delta y_2$;
dividing by δx,

$$\frac{\delta y}{\delta x} = \frac{\delta y_1}{\delta x} + \frac{\delta y_2}{\delta x},$$

or making δx (and $\therefore \delta y$) indefinitely small,

$$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{dy_2}{dx}.$$

Similarly if $y = y_1 - y_2$,

then $$\frac{dy}{dx} = \frac{dy_1}{dx} - \frac{dy_2}{dx}.$$

2°. Product: if $y = y_1 y_2$

then $$\frac{dy}{dx} = y_2 \frac{dy_1}{dx} + y_1 \frac{dy_2}{dx},$$

or in words: *the slope of* $(y_1 \times y_2) = y_2 \times$ *slope of* $y_1 + y_1 \times$ *slope of* y_2.

Here

$$y + \delta y = (y_1 + \delta y_1)(y_2 + \delta y_2).$$

$$\therefore \delta y = (y_1 + \delta y_1)(y_2 + \delta y_2) - y_1 y_2$$
$$= y_1 y_2 + y_2 \delta y_1 + y_1 \delta y_2 + \delta y_1 . \delta y_2 - y_1 y_2.$$
$$= y_2 \delta y_1 + y_1 \delta y_2 + \delta y_1 \delta y_2.$$

Hence

$$\frac{\delta y}{\delta x} = y_2 \frac{\delta y_1}{\delta x} + y_1 \frac{\delta y_2}{\delta x} + \frac{\delta y_1}{\delta x} . \delta y_2.$$

If we make δx indefinitely small, the last term becomes negligible, for $\frac{\delta y_1}{\delta x}$ remains finite while δy_2 becomes very small. So finally

$$\frac{d(y_1 y_2)}{dx} = y_2 \frac{dy_1}{dx} + y_1 \frac{dy_2}{dx}.$$

3°. Quotient; if $y = y_2/y_1$

then $$\frac{dy}{dx} = \left(y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}\right) \Big/ y_1^2,$$

Or in words: *the slope of a quotient = denominator × slope of numerator − numerator × slope of denominator, all over denominator squared.*

If $y = y_2/y_1$ then

$$\delta y = \frac{y_2 + \delta y_2}{y_1 + \delta y_1} - \frac{y_2}{y_1}$$
$$= \frac{(y_2 + \delta y_2)\, y_1 - y_2\,(y_1 + \delta y_1)}{(y_1 + \delta y_1) y_1}$$

$$= \frac{y_1\delta y_2 - y_2\delta y_1}{y_1(y_1 + \delta y_1)}.$$

Hence $\dfrac{\delta y}{\delta x} = \dfrac{y_1\dfrac{\delta y_2}{\delta x} - y_2\dfrac{\delta y_1}{\delta x}}{y_1(y_1 + \delta y_1)}$

and $\therefore \dfrac{dy}{dx} = \dfrac{y_1\dfrac{dy_2}{dx} - y_2\dfrac{dy_1}{dx},}{y_1^2}$ because δy, becomes negligibly small.

4°. FUNCTION OF A FUNCTION: $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx}$

or in words: *slope of y with respect to x = slope of y with respect to z × slope of z with respect to x.*

We now come to the case of the more complicated relations between y and x discussed in § 28 (Chap. III). Suppose y is given in terms of z so that we know $\dfrac{dy}{dz}$, and z is given in terms of x, so that we know also $\dfrac{dz}{dx}$. We want to find $\dfrac{dy}{dx}$. Dealing with finite changes in value or increments we have on introducing δz in numerator and denominator,

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x},$$

when the increments become infinitely small, each ratio becomes a differential coefficient and the formula follows at once, i.e.

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

A special case should be carefully noted. If y depends on x, then x depends on y (inverse function, §27). So, replacing z by x, instead of saying: y is a function of z which is a function of x, we might say :

x is a function of y, which is a function of x. Therefore

$$\frac{dx}{dx} = \frac{dy}{dx} \cdot \frac{dx}{dy}.$$

Now $\frac{dx}{dx}$ is 1; hence

$$1 = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

or,

$$\frac{dx}{dy} = 1\Big/\frac{dy}{dx},$$

a very useful result.

This can be seen very clearly from the graphical consideration of slopes. If a tangent to a curve intersect the y and x axes at angles α and β respectively, such angles are complementary

$$\therefore \tan\alpha = \cot\beta = 1/\tan\beta$$

$$\text{but } \tan\alpha = \frac{dx}{dy} \text{ and } \tan\beta = \frac{dy}{dx}$$

§ 31. **Fundamental Slopes.**—We shall reduce the slopes we obtain from first principles (apart from the explanatory examples in § 29) to a minimum, namely three, one for each of the classes of laws in the last chapter.

1°. $y = x^n$, where n = a whole number.

$$\frac{dy}{dx} = n \times x^{n-1}.$$

To prove this we use the *method of induction.* When $n = 1$, $y = x$ and $\delta y = \delta x$.

$$\therefore \frac{\delta y}{\delta x} = 1 \text{ and } \frac{dy}{dx} = 1$$

$$n = 2, \qquad y = x^2$$

$$\delta y = (x + \delta x)^2 - x^2 = 2x\delta x + (\delta x)^2$$

$$\therefore \frac{\delta y}{\delta x} = 2x + \delta x \text{ and } \frac{dy}{dx} = 2x + 0 = 2x$$

$$n = 3,\ y = x^3$$

we have seen in § 29 that $\dfrac{dy}{dx} = 3x^2$.

Assume the rule holds for any other whole number n. We shall prove that it holds for the next whole number $n + 1$. We assume, then, that

$$\text{if } y = x^n, \text{ then } \frac{dy}{dx} = nx^{n-1}.$$

Take $y = x^{n+1} = x^n \times x$

$$\frac{dy}{dx} = x\frac{dx^n}{dx} + x^n \frac{dx}{dx} \text{ (by § 30)}$$

$$= x \,.\, nx^{n-1} + x^n = nx^n + x^n$$

$$= (n + 1)\, x^n.$$

Then, if the rule holds for any whole number, it holds for the next. It holds for $n = 2$, ∴ for $n = 3$, ∴ for $n = 4$ and so on, and finally it holds for all whole numbers.

2°. $y = \sin x, \qquad \dfrac{dy}{dx} = \cos x$

$$y + \delta y = \sin (x + \delta x)$$

$$\delta y = \sin (x + \delta x) - \sin x$$

$$= 2 \sin \frac{\delta x}{2} \cos \left(x + \frac{\delta x}{2}\right).$$

Hence

$$\frac{\delta y}{\delta x} = \frac{2 \sin \frac{\delta x}{2}}{\delta x} \,.\, \cos \left(x + \frac{\delta x}{2}\right)$$

$$= \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} \cdot \cos\left(x + \frac{\delta x}{2}\right).$$

The first factor $\dfrac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$ approaches unity when $\dfrac{dx}{2}$ becomes indefinitely small (see § 19); the second factor approaches $\cos x$. Hence

$$\frac{dy}{dx} = \cos x.$$

3°. $y = e$, $\qquad \dfrac{dy}{dx} = e^x$.

This we have seen is a characteristic property of the exponential law (§ 26).

§ 32. **Standard Slopes.**—We now deduce a set of standard slopes from the results in §§ 26 and 27.

1°. $y = x^n$ (n = any number), $\dfrac{dy}{dx} = nx^{n-1}$

(1) let n be fractional: $n = p/q$

$$y = x^{\frac{p}{q}}.$$

Raise both sides to the q^{th} power

$$y^q = x^p.$$

Differentiate both sides with regard to x.

The left-hand side y^q is a function of y, which itself is a function of x; we therefore apply the "function of a function" rule (§ 30, 4°), and get

$$\frac{dy}{dy} \cdot \frac{dy}{dx} = \frac{dx^p}{dx}$$

$$\therefore \quad qy^{q-1} \cdot \frac{dy}{dx} = px^{p-1}.$$

So
$$\frac{dy}{dx} = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{\left(x^{\frac{p}{q}}\right)^{q-1}}$$

$$= \frac{p}{q} \frac{x^{p-1}}{x^{p-\frac{p}{q}}} = \frac{p}{q} x^{p-1-p+\frac{p}{q}}$$

$$= \frac{p}{q} x^{p-\frac{p}{q}-1} = nx^{n-1}.$$

(2) Let n be negative: $n = -m$ (m is a positive number, whole or fractional)

$$y = x^{-m}$$
$$\therefore x^m y = 1.$$

Differentiate both sides: the slope of 1 is zero, for the graph of a constant is a horizontal straight line: the left-hand side is a product (§ 30, 2°).

$$y \frac{dx^m}{dx} + x^m \frac{dy}{dx} = 0$$

$$\text{i.e. } x^m \frac{dy}{dx} = -y \frac{dx^m}{dx} = -x^{-m} \cdot mx^{m-1}$$
$$= -mx^{-1}$$

$$\therefore \frac{dy}{dx} = -mx^{-m-1} = nx^{n-1}.$$

The result then holds for all values of n.

2°. $y = \cos x$, $\frac{dy}{dx} = -\sin x$.

We can write $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, and apply the "function of a function" rule (§ 30, 4°), for $\sin\left(x + \frac{\pi}{2}\right)$ is a function of $\left(x + \frac{\pi}{2}\right)$ which is a function of x.

$$\frac{dy}{dx} = \frac{d \cos x}{dx} = \frac{d \sin\left(x + \frac{\pi}{2}\right)}{dx}$$

$$= \frac{d \sin\left(x + \frac{\pi}{2}\right)}{d\left(x + \frac{\pi}{2}\right)} \cdot \frac{d\left(x + \frac{\pi}{2}\right)}{dx}$$

$$= \cos\left(x + \frac{\pi}{2}\right) \cdot \left(\frac{dx}{dx} + \frac{d\frac{\pi}{2}}{dx}\right)$$

$$= -\sin x \, . \, (1 + 0) = -\sin x.$$

3°. $y = \tan x, \ \frac{dy}{dx} = \sec^2 x.$

Here $\tan x = \frac{\sin x}{\cos x}$ is to be regarded as a quotient (§ 30, 3°)

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d \sin x}{dx} - \sin x \frac{d \cos x}{dx}}{\cos^2 x}$$

$$= \frac{\cos x \, . \cos x - \sin x \, . \, (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

4°. $y = \sin^{-1} x, \ \frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}.$

Here we have

$$\sin y = x.$$

Applying the "function of a function" rule (§ 30, 4°)

$$\frac{d \sin y}{dy} \cdot \frac{dy}{dx} = \frac{dx}{dx}$$

$$\cos y \, . \, \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

5°. $y = \tan^{-1} x$, $\dfrac{dy}{dx} = \dfrac{1}{1 + x^2}$.

Proceeding as in 4°, we put $\tan y = x$,

$$\therefore \sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}.$$

6°. $y = \log x$, $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Again similarly

$$e^y = x$$

$$\therefore \frac{de^y}{dy} \cdot \frac{dy}{dx} = \frac{dx}{dx}$$

$$e^y \frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}.$$

7°. The student should notice the effect of *multiplying by a constant.* We will illustrate this by means of examples :—

(1) $y = a \sin x$.

It should be evident that

$$\frac{dy}{dx} = a \frac{d \sin x}{dx} = a \cos x.$$

If it is not evident, the product formula (§ 30, 2°) may be used

$$\frac{dy}{dx} = \frac{d}{dx} a \,.\, \sin x$$

$$= \sin x \frac{da}{dx} + a \frac{d \sin x}{dx}$$

$$= \sin x \times 0 + a \cos x$$

$$= a \cos x.$$

(2) $$y = \sin bx.$$

Regard this as a "function of a function".

$$\frac{dy}{dx} = \frac{d \sin bx}{dx} = \frac{d \sin (bx)}{d(bx)} \cdot \frac{d(bx)}{dx}$$

$$= \cos bx \times b = b \cos bx.$$

The following list of standard slopes should be remembered. The student should work out for himself those which have not been obtained in the text.

LIST OF STANDARD SLOPES OR DIFFERENTIAL COEFFICIENTS.

1.	$y =$ constant	$\frac{dy}{dx} = 0$
2.	$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$
3.	$\sin x$	$\cos x$
4.	$\cos x$	$-\sin x$
5.	$\tan x$	$\sec^2 x$
6.	$\operatorname{cosec} x$	$-\operatorname{cosec}^2 x \cos x$
7.	$\sec x$	$\sec^2 x \sin x$
8.	$\cot x$	$-\operatorname{cosec}^2 x$
9.	$\sin^{-1} x$	$1/\sqrt{1 - x^2}$
10.	$\cos^{-1} x$	$-1/\sqrt{1 - x^2}$
11.	$\tan^{-1} x$	$1/(1 + x^2)$
12.	e^x	e^x
13.	$\log x$	$1/x.$

§ 33. **Implicit Relations.**—We have so far considered cases where the dependent quantity y is given directly (*explicitly*) in terms of x, as in

$$y = \frac{1}{1 + x}.$$

We frequently have y given in terms of x in an indirect (*implicit*) way, as in

1°. $x^2 + 2xy + 1 + y = 0.$
2°. $y^2 + 3xy + 2x^2 = 0.$
3°. $e^{xy} = x + y.$

As before y can be determined when x is known, but only by solving an equation. In 1° we get the explicit form at once

$$y = -\frac{x^2+1}{2x+1}.$$

In 2° we find (on solving the quadratic equation in y) our relation is equivalent to *two explicit relations*

$$y = -x \text{ and } y = -2x.$$

In 3° we *cannot find* the equivalent explicit relations.

In such cases it is still often necessary and easy to find $\frac{dy}{dx}$. To do so we only have to differentiate the equation through, *remembering that y depends upon x* (is a function of x).

Taking 2°

$$2y\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} + 4x = 0.$$

Hence
$$\frac{dy}{dx} = -\frac{3y+4x}{2y+3x}.$$

It is seen that in this case both x and y appear in the expression for the slope. This as a rule is no disadvantage, and even in cases where we can replace y by its expression in x (we could in this example), it is generally not worth while doing so.

In example 3° we have similarly

$$\frac{de^{(xy)}}{d(xy)}\,\frac{d(xy)}{dx} = \frac{dx}{dx} + \frac{dy}{dx},$$

$$\text{i.e. } e^{xy}\left(y + x\,\frac{dy}{dx}\right) = 1 + \frac{dy}{dx}.$$

Hence

$$\frac{dy}{dx} = \frac{1 - ye^{xy}}{xe^{xy} - 1}.$$

§ 34. **Successive Differentiation.**—Having found the slope of any expression, say x^n, we have another expression, nx^{n-1}, of which we can find the the slope in turn (§ 5). And this process may be carried out as often as we please. These successive differential cœfficients are written

$$\frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \frac{d^3y}{dx},\ \text{etc.}$$

So that

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = \left(\frac{d}{dx}\right)^2 y, \text{ etc.}$$

Thus if

1° $$y = x^n,$$

then

$$\frac{dy}{dx} = nx^{-1},\ \frac{d^2y}{dx^2} = n\,(n-1)\,x^{n-2},$$

$$\frac{d^3y}{dx^3} = n(n-1)\,(n-2)\,x^{n-3},$$

$$\frac{d^4y}{dx^4} = n\,(n-1)\,(n-2)\,(n-3)\,x^{n-4},$$

and so on.

Note that the nth slope when n is integral,

$$\frac{d^ny}{dx^n} = n\,(n-1)\,(n-2)\ \ .\ \ .\ \ .\ \ 3.\ 2.\ 1.$$

is a constant, and all higher slopes vanish.

Other examples :—

2°.

$$y = \log x$$

$$\frac{dy}{dx} = \frac{1}{x}, \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}, \quad \frac{d^3y}{dx^3} = +\frac{2}{x^3},$$

$$\frac{d^4y}{dx^4} = -\frac{2 \,.\, 3}{x^4} = -\frac{6}{x^4}, \text{ and so on.}$$

3°.

$$y = e^x$$

$$\frac{dy}{dx} = \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = \ldots = e^x = y.$$

4°.

$$y = \sin x$$

$$\frac{dy}{dx} = \cos x = \sin\left(x + \frac{\pi}{2}\right)$$

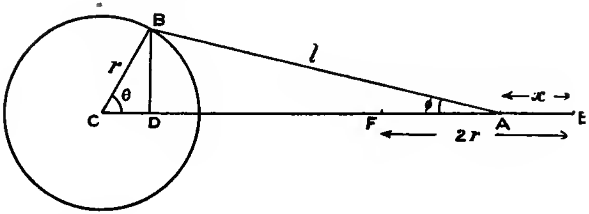

Fig. 44.

$$\frac{d^2y}{dx^2} = \cos\left(x + \frac{\pi}{2}\right) = \sin\left(x + 2.\,\frac{\pi}{2}\right) = -\sin x$$

$$\frac{d^3y}{dx^3} = \sin\left(x + 3.\,\frac{\pi}{2}\right) = -\cos x$$

$$\frac{d^4y}{dx^4} = \sin\left(x + 4.\,\frac{\pi}{2}\right) = \sin x.$$

§ 35. Example of Steam-Engine Mechanism.—*As a useful example involving fairly complicated differentiation, take the case of the steam engine mechanism that we have already considered* (p. 13).

The crank CB revolves around the point C with uniform velocity and the cross-head A reciprocates between the points EF. To find the velocity and acceleration of the cross-head.

Let the cross-head be at a distance x from E.

Velocity.—Now velocity = rate of change of position, i.e. $\frac{dx}{dt}$

$$\begin{aligned} x &= \mathrm{EC} - \mathrm{AC} \\ &= l + r - l\cos\phi - r\cos\theta. \end{aligned}$$

Let $l = kr$,

then $x = r\,\{k\,(1 - \cos\phi) + (1 - \cos\theta)\}$ (1)

$$\begin{aligned} \therefore v &= \frac{dx}{dt} \\ &= r\left\{k.\frac{d(1-\cos\phi)}{d\phi}\cdot\frac{d\phi}{dt}\right. \\ &\qquad \left. + \frac{d(1-\cos\theta)}{d\theta}\cdot\frac{d\theta}{dt}\right\}^* \\ &= r\left\{k\sin\phi\frac{d\phi}{dt} + \sin\theta\frac{d\theta}{dt}\right\} \qquad (2) \end{aligned}$$

Now $\frac{d\theta}{dt}$ = rate of change of angle = angular velocity = $\omega = 2\pi n$, n being the number of revolutions per second.

Also $l\sin\phi = \mathrm{BD} = r\sin\theta$,

$$\therefore k\sin\phi = \sin\theta, \qquad . \quad . \quad (3)$$

$\therefore$ differentiating we get

$$k\cos\phi\frac{d\phi}{dt} = \cos\theta\frac{d\theta}{dt} \qquad . \quad . \quad (4)$$

Substituting (3) and (4) in (2),

$$v = r\left(\frac{k\sin\phi\cos\theta}{k\cos\phi}\cdot\frac{d}{dt} + \sin\theta\frac{d\theta}{dt}\right)$$

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* This is an example of "Function of a function" (pp. 75 and 83).

$$= r\frac{d\theta}{dt}\left(\frac{\sin\theta\cos\theta}{k\cos\phi} + \sin\theta\right)$$

$$= 2\,\pi rn\sin\theta\left(1 + \frac{\cos\theta}{k\cos\phi}\right). \quad . \quad (5)$$

From (3) $\qquad k^2\sin^2\phi = \sin^2\theta$

i.e. $\qquad k^2(1 - \cos^2\phi) = \sin^2\theta$

i.e. $\qquad k^2\cos^2\phi = k^2 - \sin^2\theta$

or $k\cos\phi = \sqrt{k^2 - \sin^2\theta}$

$$\therefore\ v = 2\pi rn\sin\theta\left(\frac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}} + 1\right) \quad . \quad . \quad (6)$$

Acceleration.—Now acceleration = rate of change of velocity

$$= a = \frac{dv}{dt} = \frac{dv}{d\theta}\cdot\frac{d\theta}{dt} = 2\pi n.\ \frac{dv}{d\theta} \quad . \quad . \quad (7)$$

$$\therefore\ a = 4\pi^2n^2r\left[\sin\theta\ .\ \frac{d\left\{\dfrac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}} + 1\right\}}{d\theta}\right.$$

$$\left. + \left(\frac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}} + 1\right)\frac{d\sin\theta}{d\theta}\right] \quad . \quad . \quad (8)$$

Now $\qquad \dfrac{d\left\{\dfrac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}} + 1\right\}}{d\theta}$

$$= \frac{\sqrt{k^2 - \sin^2\theta}.\ \dfrac{d\cos\theta}{d\theta} - \cos\theta\ \dfrac{d\sqrt{k^2 - \sin^2\theta}}{d\theta}}{(k^2 - \sin^2\theta)} + 0$$

$$= \frac{-\sin\theta\sqrt{k^2 - \sin^2\theta} - \cos\theta\ .\ \frac{1}{2}\left(\sqrt{k^2 - \sin^2\theta}\right)^{-\frac{1}{2}}.(-2\sin\theta.\cos\theta)^{*}}{(k^2 - \sin^2\theta)}$$

$$= \frac{\dfrac{\sin\theta\cos^2\theta}{\sqrt{k^2 - \sin^2\theta}} - \sin\theta\sqrt{k^2 - \sin^2\theta}}{(k^2 - \sin^2\theta)}$$

$$^{*}\frac{d\sin^2\theta}{d\theta} = 2\sin\theta\frac{d\sin\theta}{d\theta} = 2\sin\theta\cos\theta.$$

$$= \frac{\sin\theta\,\{\cos^2\theta - (\sqrt{k^2 - \sin^2\theta})^2\}}{(k^2 - \sin^2\theta)^{\frac{3}{2}}}$$

$$= \frac{\sin\theta\,(\cos^2\theta - k^2 + \sin^2\theta)}{(k^2 - \sin^2\theta)^{\frac{3}{2}}} = \frac{\sin\theta\,(1 - k^2)}{(k^2 - \sin^2\theta)^{\frac{3}{2}}}.$$

Putting this value in (8) we get

$$a = 4\pi^2 n^2 r \left\{\frac{\sin^2\theta\,(1 - k^2)}{(k^2 - \sin^2\theta)^{\frac{3}{2}}} + \cos\theta\left(1 + \frac{\cos\theta}{\sqrt{k^2 - \sin^2\theta}}\right)\right\}$$

$$= 4\pi^2 n^2 r \left\{\cos\theta + \frac{\sin^2\theta\,(1 - k^2) + \cos^2\theta\,(k^2 - \sin^2\theta)}{(k^2 - \sin^2\theta)^{\frac{3}{2}}}\right\}$$

$$= 4\pi^2 n^2 r \left\{\cos\theta + \frac{k^2\,(\cos^2\theta - \sin^2\theta) + \sin^2\theta\,(1 - \cos^2\theta)}{(k^2 - \sin^2\theta)^{\frac{3}{2}}}\right\}$$

$$= 4\pi^2 n^2 r \left\{\cos\theta + \frac{k^2\cos 2\theta + \sin^4\theta}{(k^2 - \sin^4\theta)^{\frac{3}{2}}}\right\} \qquad (8)$$

Equation (8) therefore gives the acceleration of the cross-head and therefore that of the piston for any angular position of the crank.

EXERCISES 4.

When the student has grasped the principles of differentiation he requires considerable practice in carrying out the operation in definite exercises and so should work carefully through the following :—

Find the differential coefficients with regard to x of :—

1. $5x^2$.
2. $6x^2 - 6x + 12$.
3. $4 + 11x - 2x^2$.
4. $\dfrac{2}{x^3}$.
5. $\dfrac{4}{\sqrt{x^5}}$.
6. $x + \dfrac{1}{x}$.
7. $4x + \dfrac{1}{x^2}$.
8. $(4x + 1)^5$.
9. $(3 - x)^4$.

10. $\dfrac{1}{(2x + 4)^2}$.
11. $\dfrac{1}{(x + 1)^2}$.
12. $\dfrac{(x + 1)^2}{x^2 + 1}$.
13. $\dfrac{2x^4}{16 - x^2}$.
14. $\sin \sqrt{x^2 + 3}$.
15. $\sin^2 x$.
16. $\sin 2x \cos x$.
17. $x \sin x$.
18. $x \log_e x$.
19. $\log_e \sin x$.
20. $\sin^{-1} (1 - x)$.
21. $\log_e \dfrac{1 - x}{1 + x}$.
22. $\log_e \{x + \sqrt{x^2 + 1}\}$.
23. $\log_e \{x + \sqrt{x^2 - 1}\}$.
24. $y = \frac{1}{2} (e^x - e^{-x})$.
25. $\sqrt{\left(\dfrac{1 + x^2}{1 - x^2}\right)}$.
26. $\sin x^2$.
27. $\sin^{-1} 2x$.
28. $\tan^{-1} x^2$.
29. $\cos^{-1} \dfrac{1}{x}$.
30. $2 \log_{10} x$.
31. $\cot \log_e x$.
32. $\log_e \cot x$.
33. e^{-x}.
34. e^{-x^2}.
35. $e^{-\frac{x}{10}} \sin 3x$.

36. If $s = a \cos (mt + b)$, find $\dfrac{d^2s}{dt^2}$.

37. If $y = \mathrm{A} \sin mx + \mathrm{B} \sin mx$, find $\dfrac{d^2y}{dt^2}$.

38. If $y = x^3 \log_e \dfrac{x}{a}$, find $\dfrac{d^4y}{dx^4}$.

39. If $y = 3x^3 + 4x^2 - 3x + 6$, find $\dfrac{d^2y}{dx^2}$.

40. If $y = \sin^2 x$, find $\dfrac{d^2y}{dx^2}$.

41. In the steam-engine show that when the piston is at its mid point in the forward motion, the crank angle is $\cos^{-1} \dfrac{1}{2k}$ and find the acceleration and velocity in that position.

ANSWERS TO EXERCISES.

4.

1. $10x$. 2. $12x - 6$. 3. $11 - 4x$. 4. $\frac{-6}{x^4}$.

5. $-\frac{10}{\sqrt{x^7}}$. 6. $1 - \frac{1}{x^2}$. 7. $4 - \frac{2}{x^3}$.

8. $20(4x+1)^4$. 9. $-4(3-x)^3$. 10. $-\frac{4}{(2x+4)^3}$.

11. $\frac{-2}{(x+1)^3}$ 12. $\frac{2(1-x^2)}{(x^2+1)^2}$. 13. $\frac{128x^3 - 4x^5}{(16-x^2)^2}$.

14. $\frac{x \cos \sqrt{x^2+3}}{\sqrt{x^2+3}}$ 15. $\sin 2x$.

16. $2 \cos 2x \cos x - \sin 2x \sin x$.

17. $\sin x + x \cos x$. 18. $1 + \log x$. 19. $\cot x$.

20. $\frac{-1}{\sqrt{2x - x^2}}$. 21. $\frac{-2}{1 - x^2}$. 22. $\frac{1}{\sqrt{1 + x^2}}$.

23. $\frac{1}{\sqrt{x^2-1}}$. 24. $\frac{1}{2}(e^x + e^{-x})$. 25. $\frac{2x}{\sqrt{(1+x^2)(1-x^2)^3}}$.

26. $2x \cos x^2$. 27. $\frac{2}{\sqrt{1-4x^2}}$. 28. $\frac{2x}{(1+x^4)}$.

29. $\frac{1}{x\sqrt{x^2-1}}$. 30. $2 \log_{10} e . \frac{1}{x}$.

31. $-\frac{1}{x} \operatorname{cosec}^2 \log_e x$. 32. $-\frac{1}{\sin x \cos x}$.

33. $-e^{-x}$ 34. $2xe^{-x^2}$.

35. $e^{-\frac{x}{10}}\left(-\frac{\sin 3x}{10} + 3 \cos 3x\right)$.

36. $-m^2 s$. 37. $-m^2 y$. 38. $\frac{6}{x}$. 39. $18x + 8$.

40. $2 \cos 2x$. 41. $v = \frac{2\pi n r\sqrt{4k^2-1}}{2k(2k^2-1)}$,

$$a = \frac{(2\pi n)^2 (4k^4 - 6k^2 + 1)}{(2k^2 - 1)^3}.$$

CHAPTER V.

APPLICATIONS OF DIFFERENTIATION.

CRITICAL VALUES—MAXIMA AND MINIMA.

§ 36. Let us return to the discussion in § 3. If the figure represents the graph corresponding to some law, then the point

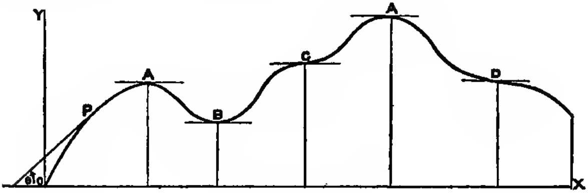

FIG. 45.

A *is a maximum, since y rises to a greatest value and then falls*;

B *is a minimum, since y falls to a least value and then rises.*

At each of these points the curve is for a moment horizontal, and if θ is the inclination of the tangent at any point to the x-axis, we shall have at A and B

$$\text{slope} = \frac{dy}{dx} = \tan\theta = 0.$$

But $\frac{dy}{dx}$ vanishes also at such a point as C where the curve rises and becomes horizontal only to rise again.

The points

C *and* D *are points of inflexion with horizontal tangents,*

and all such points as A, B, C, D *are critical points.*

We now have to deal with this problem :—

Given a law in the shape of a formula to find its critical points and to discuss their nature.

In the first place all the critical values are obtained by writing

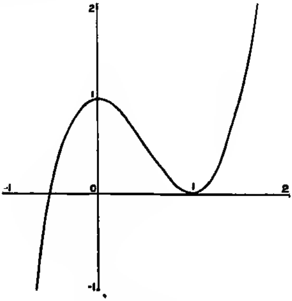

FIG. 46.

$$\frac{dy}{dx} = 0;$$

for example, if

$$y = 2x^3 - 3x^2 + 1, \quad . \quad . \quad . \quad (1)$$

then the critical values are given by

$$\frac{dy}{dx} = 6x^2 - 6x = 0$$

$$\text{i.e. } x^2 - x = 0,\ x(x-1) = 0$$

$$x = 0 \text{ and } x = 1;$$

the critical values of y are

(1) for $x = 0$, $y = 0 + 0 + 1 = 1$

(2) for $x = 1$, $y = 2.1 - 3.1 + 1 = 0$.

Having obtained the critical values, it is necessary to find out whether they are maxima or minima or only points of inflexion.

Drawing the derived curve of Fig. 45, we note (Fig. 47) that the slope which was positive before the point A, vanishes at A (M) and becomes negative. So

At a maximum the slope is diminishing, and the slope curve is descending.

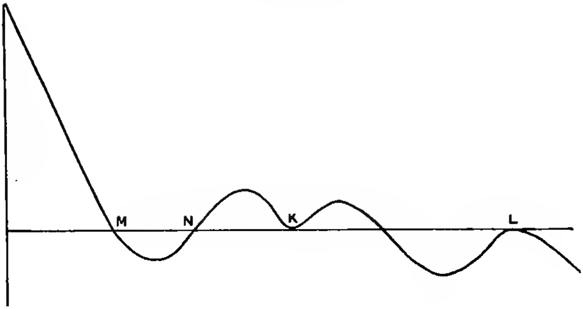

FIG. 47.

The slope of the slope curve is negative, i.e.

$$\frac{d^2y}{dx^2} \textit{ is negative at a maximum.}$$

Similarly

$$\frac{d^2y}{dx^2} \textit{ is positive at a minimum.}$$

At a point of inflexion such as C (K) (Fig. 45), the slope which was positive diminishes down to zero, but *then it increases* immediately after. We must therefore have a minimum on the slope curve; similar remarks apply to a point such as D (L), and hence

$\frac{d^2y}{dx^2} = 0$ *at a point of inflexion (with a horizontal tangent).*

For the expression (1)

$$\frac{d^2y}{dx^2} = 12x - 6.$$

Hence

(1) for $x = 0, \frac{d^2y}{dx^2} = -6$, maximum.

(2) for $x = 1, \frac{d^2y}{dx^2} = 12 - 6 = +6$, minimum.

As an example of a point of inflexion, take

$$y = x(x-1)^3;$$

the critical values are the roots of

$$\frac{dy}{dx} = (x-1)^3 + 3x(x-1)^2 = 0$$

$$\text{i.e. } (x-1)^2(x-1+3x) = 0$$

$$(x-1)^2(4x-1) = 0$$

$$\therefore x = 1 \text{ and } x = \tfrac{1}{4}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\{(x-1)^2(4x-1)\}$$

$$= 2(x-1)(4x-1) + 4(x-1)^2$$

$$= 6(x-1)(2x-1)$$

$x = 1$ gives $\frac{d^2y}{dx^2} = 0$, point of inflexion.

$x = +\frac{1}{4}$ gives $\frac{d^2y}{dx^2} = \frac{9}{4}$, minimum.

It should be noted that in general a maximum value of y is not *the greatest value*, for after rising to a maximum and falling, the curve may rise again beyond the maximum. An inspection of Fig. 45 will show further that maxima and minima must generally occur alternately (always when there is no break in the curve).

Note.—The above investigation is not complete, although quite exact as far as it goes. If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at a certain point, this point is generally a point of inflexion, but it may exceptionally still be a maximum or a minimum. The complete rule is: *At a critical point let*

$$\frac{d^2y}{dx^2} = 0, \ \frac{d^3y}{dx^3} = 0, \text{ etc.}$$

the *first* slope which does not vanish being

$$\frac{d^ry}{dx^r}$$

then

(1) if r is odd, we have a point of inflexion.

(2) if r is even, then

if $\frac{d^ry}{dx^r} < 0$ we have a maximum,

if $\frac{d^ry}{dx^r} > 0$ we have a ~~maximum~~. minimum

For instance, consider

$$y = (x - 1)^4$$

$$\frac{dy}{dx} = 4\ (x - 1)^3: \text{ critical value is } x = 1;$$

$$\text{for } x = 1, \ \frac{d^2y}{dx^2} = 4.3\ (x - 1)^2 = 0$$

$$\frac{d^3y}{dx^3} = 4.3.2\ (x - 1) = 0$$

$$\frac{d^4y}{dx^4} = 4.3.2.1 = +\ 24.$$

Hence $x = 1$ is a minimum.

EXERCISES 5A.

Simple Problems on Maxima and Minima.

1. Find the critical value of $8 + 15x - 3x^2$. Is it a maximum or a minimum?

2. Find the maximum and minimum of

$$x^3 - 8x + 6.$$

3. Find the maximum and minimum values of $x(9 - x^2)$.

4. What is the maximum value of $(2x+3)(5-x)$?

5. Prove that the rectangle of least perimeter for a given area is the square.

6. Find the area of the greatest rectangle that can be inscribed in a given triangle.

7. Find the values of x for a maximum value of the graph $y = xe^{-x}$.

8. Show that $\sin x(1 + \cos x)$ is a maximum for $x = 60°$.

9. Find the minimum value of $a \cot \theta + b \tan \theta$ between $\theta = 0$ and $\theta = \frac{\pi}{2}$.

10. Find the values of θ which make $\cos \theta + \sin \theta$ a maximum.

§ 37. **Worked Problems on Maxima and Minima.**—The following examples will show some of the many applications to problems of engineering interest of the above treatment of maxima and minima.

1. *A man* A *is* 5 *miles out at sea in a boat opposite a point* B *and wishes to reach a point* C *on the shore at a distance of* 8 *miles from* B. *If he can walk* 4 *miles an hour and row* 3 *miles an hour, find at what point he should land in order to reach* C *in the shortest possible time.*

Suppose that he lands at a point D at distance x from B.

Then the distance that he rows

$$= \text{AD} = \sqrt{25 + x^2} \text{ miles}$$

and the distance that he walks $= \text{DC} = 8 - x$ miles.

$$\therefore \text{Time taken} = t = \frac{\text{AD}}{3} + \frac{\text{DC}}{4}$$

$$= \frac{\sqrt{25 + x^2}}{3} + \frac{(8 - x)}{4} \quad . \quad . \quad . \quad (1)$$

$$\therefore \frac{dt}{dx} = \frac{d\left(\frac{\sqrt{25 + x^2}}{3}\right)}{dx} + \frac{d\left(\frac{8 - x}{4}\right)}{dx}$$

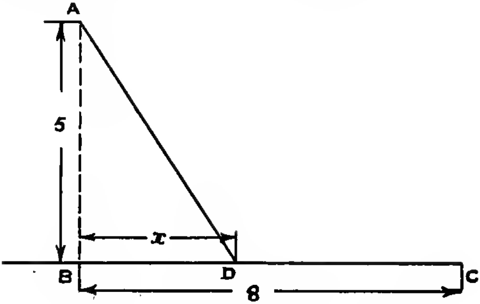

FIG. 48.

$$= \tfrac{1}{3}\,\frac{d\,(25 + x^2)^{\frac{1}{2}*}}{dx} + \tfrac{1}{4}\,\frac{d\,(8 - x)}{dx}$$

$$= \tfrac{1}{3}\,\frac{(25 + x^2)^{\frac{1}{2} - 1}}{2} \,.\, d\left(\frac{25 + x^2}{dx}\right) + \tfrac{1}{4}\,(-\,1)$$

$$= \frac{(25 + x^2)^{-\frac{1}{2}}}{2.3} \,.\, 2x - \tfrac{1}{4}$$

$$= \frac{x}{3\sqrt{25 + x^2}} - \tfrac{1}{4} \quad . \quad . \quad . \quad (2)$$

For a minimum value $\frac{dt}{dx} = 0$

---

* This is an example of $\frac{dy^n}{dx}$, cf. p. 84.

$$\therefore \frac{x}{3\sqrt{25+x^2}} - \frac{1}{4} = 0$$

$$\text{i.e. } 4x = 3\sqrt{25+x^2}$$

$$\text{(squaring) } 16x^2 = 225 + 9x^2$$

$$x^2 = \frac{225}{7}$$

$$x = \frac{15}{\sqrt{7}} = \frac{15\sqrt{7}}{7} = \underline{5{\cdot}7 \text{ miles, Ans.}}$$

(ignoring the negative root).

The corresponding time is 3·102 hours.

We have not yet proved that this is the minimum value but it is easy to do so as follows:—

If $x = 0$, $t = 3{\cdot}67$ hours
„ $x = 5{\cdot}7$, $t = 3{\cdot}102$ „
„ $x = 8$, $t = 3{\cdot}145$ „

This is the whole range of x and clearly t must be a minimum.

The other and more reliable test is to find $\frac{d^2t}{dx^2}$. This is not often necessary in practical cases because there is seldom any doubt, but we will do it as an example

$$\frac{d^2t}{dx^2} = \frac{d\left(\frac{x}{3\sqrt{25+x^2}}\right)}{dx} - \frac{d\left(\frac{1}{4}\right)}{dx}$$

$$= \frac{1}{3}\left\{\frac{\sqrt{25+x^2}\,.\,\frac{dx}{dx} - \frac{x\,d\left(\sqrt{25+x^2}\right)}{dx}}{\left(\sqrt{25+x^2}\right)^2}\right\} - 0$$

$$= \frac{1}{3}\left(\frac{\sqrt{25+x^2} - x^2\left(25+x^2\right)^{-\frac{1}{2}}}{25+x^2}\right)$$

$$= \frac{25+x^2-x^2}{3\left(25+x^2\right)^{\frac{3}{2}}} = \frac{25}{3\left(25+x^2\right)^{\frac{3}{2}}}.$$

This is positive for $x = 15/\sqrt{7}$ (and in fact for all values of x: note that there is no question as to the sign of the root, as we have adopted and retained the positive determination in the problem). Hence the point is a minimum (§ 36, p. 101).

2. (C) *Find the position of a short uniformly distributed load which will give the maximum bending moment at any point of a simply supported girder.*

Suppose that the centre of the load, which is of intensity p and length l (Fig. 49), and of total amount $W = pl$, has reached a point D.

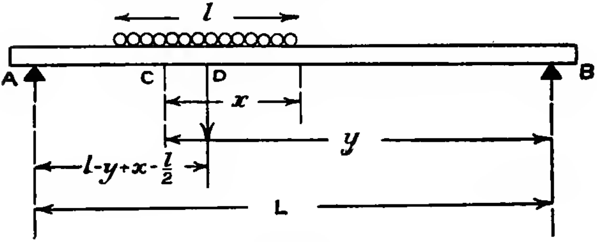

FIG. 49.

Let C be the point at which the bending moment is required, and let the front of the load be at a distance x from C and let the support B be at a distance y from C. Let R_A, R_B be the reactions at A and B, and let AB = L.

$$\text{Then } R_B = \frac{W \cdot AD}{L} = \frac{pl}{L}\left(L - y + x - \frac{l}{2}\right).$$

$$\text{Bending moment at C} = M_C = R_B \cdot y - \frac{px^2}{2}$$

$$= \frac{ply}{L}\left(L - y + x - \frac{l}{2}\right) - \frac{px^2}{2} \quad . \quad . \quad . \quad (1)$$

We want to know for what value of x this bending moment will be a maximum. We therefore treat x

as the variable and put $\frac{dM_C}{dx} = 0$, regarding y as a constant.

$$\frac{dM_C}{dx} = \frac{ply}{L}\frac{d\left(L - y + x - \frac{l}{2}\right)}{dx} - \frac{p}{2}\frac{dx^2}{dx}$$

$$= \frac{ply}{L}(0 - 0 + 1 - 0) - \frac{p}{2}\cdot 2x$$

$$= \frac{ply}{L} - px \quad . \quad . \quad . \quad (2)$$

If this $= 0$, $\frac{ly}{L} = x$

or $\frac{x}{l} = \frac{y}{L}$.

We therefore get the rule : *The B.M. at any point is a maximum when the load is in such a position that the given point divides the load in the same ratio as it divides the span.*

If there is any doubt as to whether the result is a maximum or a minimum, and in practical problems there is seldom any doubt, we find $\frac{d^2M_C}{dx^2}$.

From (2) $\frac{d^2M_C}{dx^2} = 0 - p$.

This is negative, therefore the B.M. is a maximum.

3. (M) *The horse-power that can be transmitted by a belt one square inch in section running at V feet per minute is given by the formula*

$$H.P. = \frac{7V}{891} - \frac{V^3}{12{,}400 \times 10^6}.$$

What speed will give the maximum horse-power and what will be the value of the maximum horse-power ?

In this case V is the variable, so we put

$$\frac{d\,(\text{H.P.})}{d\text{V}} = 0.$$

$$\frac{d\,(\text{H.P.})}{d\text{V}} = \frac{7}{891} - \frac{3\text{V}^2}{12{,}400 \times 10^6} = 0$$

$$\therefore\ \text{V}^2 = \frac{12{,}400 \times 10^6 \times 7}{3 \times 891}$$

$$\text{V} = 5700 \text{ feet per minute.}$$

Putting this value for V in the formula for horsepower, we get

$$\text{max. H.P.} = 29{\cdot}9$$

4. (E) *Find the relative values of the internal and external resistance of an electric battery to obtain the greatest power therefrom.*

If R_i and R_e are the internal and external resistances respectively, C the current and V the voltage, we have

$$C = \frac{V}{R_i + R_e}.$$

The power given out = W = C^2R_e

$$\therefore\ W = \frac{V^2R_e}{(R_i + R_e)^2}$$

regarding R_e as a variable we put $\frac{dW}{dR_e} = 0$.

Then $\frac{dW}{dR_e}$

$$= V^2\left[(R_i + R_e)^2 . \frac{dR_e}{dR_e} - R_e\frac{d\,(R_i + R_e)^2}{dR_e}\right] \div (R_i + R_e)^4$$

$= 0.$

$$\text{i.e. } (R_i + R_e)^2 - R_e \,.\, 2\,(R_i + R_e)\,1 = 0$$

$$(R_i + R_e)\,(R_i - R_e) = 0$$

$$\text{i.e. } R_i = R_e,$$

on rejecting the solution $R_i + R_e = 0$.

$\therefore$ *the maximum power is obtained when the external resistance is equal to the internal resistance.*

Alternative solution.—In problems of this kind when the denominator is more complicated than the numerator, we can often find the minimum value of the quantity more easily by finding the maximum of the reciprocal and vice versa,

$$\text{e.g. } \frac{1}{W} = \frac{1}{V^2}\left(\frac{R_i^2 + 2R_iR_e + R_e^2}{R_e}\right)$$

$$= \frac{1}{V^2}\left(R_i^2R_e^{-1} + 2R_i + R_e\right)$$

$$\frac{d\frac{1}{W}}{dR_e} = \frac{1}{V^2}\left(\frac{-R_i^2}{R_e^2} + 1\right) = 0$$

$$\text{i.e. } R_e^2 - R_i^2 = 0 \text{ or } R_i = R_e.$$

To test whether this gives a maximum, differentiate again, thus getting

$$\frac{d\frac{1}{V^2}\left(1 - \frac{R_i^2}{R_e^2}\right)}{dR_e} = \frac{1}{V^2}\{0 - R_i^2(-2)/R_e^{-3}\}$$

$$= \frac{2R_i^2}{R_e^3V^2}.$$

This is positive, therefore the above value is a minimum for $\frac{1}{W}$ or a maximum for W.

5. *The regulations of the parcel post stipulate that a parcel must not exceed six feet in length and girth combined. What is the size of the parcel of the greatest volume that can be sent?*

Let l be the length in feet of the parcel, then the girth or perimeter of the parcel will be $(6 - l)$ feet.

Now if A is the area of the cross section, we have $V = A \cdot l$.

For a given perimeter the circle has a larger area than any other figure,* therefore V will be greatest for a given length for a right cylinder.

$$\text{Now } A = \pi r^2$$

$$\text{and } r = \frac{\text{perimeter}}{2\pi} = \frac{6 - l}{2\pi}.$$

$$\therefore A = \frac{(6 - l)^2}{4\pi}.$$

$$\therefore V = \frac{l(6 - l)^2}{4\pi}.$$

$$\frac{dV}{dl} = \frac{1}{4\pi}\frac{d(36l - 12l^2 + l^3)}{dl} = 0$$

$$\text{i.e. } 36 - 24l + 3l^2 = 0$$

$$12 - 8l + l^2 = 0$$

$$(l - 6)(l - 2) = 0$$

$$\therefore l = 2 \text{ or } 6.$$

6 is clearly the length for minimum volume, since the girth would then be 0.

Therefore the length of the parcel is 2 feet and its diameter $= \frac{4}{\pi} = 1{\cdot}273$ *feet.*

The volume will be $\pi r^2 l = \pi \times \left(\frac{2}{\pi}\right)^2 \times 2 = \frac{8}{\pi} = 2{\cdot}546$ cubic feet.

6. (C) *If the velocity of flow through a pipe is proportional to* $\sqrt{m}$ *for a given fall of head and m is*

---

* The mathematical proof of this depends upon the Calculus of Variations and cannot be given here. The fact will be familiar. For example, if the inner tube of a bicycle tyre is inflated the pressure of the air inside will make the internal volume, and therefore the cross section, as large as possible before stretching the rubber; the cross section becomes circular. A sphere is the surface of given area which contains the greatest volume, as may be inferred from the shape of a soap bubble.

$\frac{\textit{area of water flowing}}{\textit{wetted perimeter of pipe}}$, *find how full a circular pipe should run to give a maximum discharge.*

Now flow per second = velocity × area

$$= q = k\sqrt{m} \times \text{A}$$

$$= k\sqrt{\frac{\text{A}}{\text{P}}} \times \text{A}$$

where P = wetted perimeter = ADB (Fig. 50 and k is a constant coefficient.

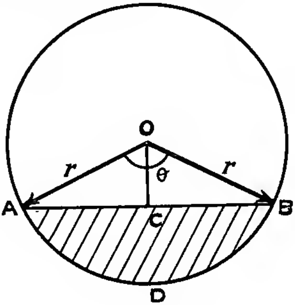

FIG. 50.

Now $\text{P} = r\theta$

$$\text{A} = \text{area sector OADB} - \text{area } \Delta\text{OAB}$$

$$= \frac{r^2\theta}{2} - \tfrac{1}{2}r^2 \sin\theta$$

(Since the area of a triangle = ½ product of two sides × sine of contained angle.)

$$= \frac{r^2}{2}(\theta - \sin\theta)$$

$$\therefore\ q = k\sqrt{\frac{r^2 \cdot (\theta - \sin\theta)}{2r\theta}} \cdot \frac{r^2}{2}(\theta - \sin\theta).$$

This will be a maximum when

$$\sqrt{1 - \frac{\sin\theta}{\theta}} \cdot (\theta - \sin\theta)$$

is a maximum,

$$\text{i.e. when } \theta\left(1-\frac{\sin\theta}{\theta}\right)^{\frac{3}{2}} \text{ is a maximum,}$$

$$\text{i.e. when } \frac{d\left\{\theta\left(1-\frac{\sin\theta}{\theta}\right)^{\frac{3}{2}}\right\}}{d\theta}^{*} = 0,$$

$$\text{i.e. } \theta\frac{d\left(1-\frac{\sin\theta}{\theta}\right)^{\frac{3}{2}}}{d\theta} + \left(1-\frac{\sin\theta}{\theta}\right)^{\frac{3}{2}} \cdot \frac{d\theta}{d\theta} = 0$$

$$\text{i.e. } \theta \cdot \tfrac{3}{2} \cdot \left(1-\frac{\sin\theta}{\theta}\right)^{\frac{1}{2}} \cdot \frac{d\left(1-\frac{\sin\theta}{\theta}\right)}{d\theta} + \left(1-\frac{\sin\theta}{\theta}\right)^{\frac{3}{2}} = 0$$

$$\text{i.e. } \left(1-\frac{\sin\theta}{\theta}\right)^{\frac{1}{2}}\left\{\frac{3\theta}{2}\left(\frac{-\theta\cos\theta+\sin\theta}{\theta^2}\right) + \left(1-\frac{\sin\theta}{\theta}\right)^{\frac{3}{2}}\right\} = 0$$

or rejecting the factor $\left(1-\frac{\sin\theta}{\theta}\right)^{\frac{1}{2}}$

$$\theta - \sin\theta - \frac{3\,\theta\cos\theta}{2} + \frac{3\sin\theta}{2} = 0;$$

$$\text{i.e. } \theta = \frac{-\sin\theta}{2-3\cos\theta}.$$

We cannot solve this equation by direct means so we proceed by trial, tabulating as follows (see next page).

It is fairly clear that the pipe must be nearly full to give a maximum discharge, so we will start with $\theta = 300°$. We pass over the choice of trial values necessary to approach the neighbourhood of the correct value, and give a few trials close round it.

---

* This is an example of the differentiation of a product (cf. p. 82).

$\theta°$.	$\sin\theta$.	$\cos\theta$.	$2-3\cos\theta$.	$\frac{-\sin\theta}{2-3\cos\theta}$.	θ radians.
300	$-\cdot 8660$	$\cdot 5000$	$\cdot 5000$	$1\cdot 74$	$5\cdot 238$
305	$-\cdot 8192$	$\cdot 5736$	$\cdot 2792$	$2\cdot 93$	$5\cdot 233$
307	$-\cdot 7986$	$\cdot 6018$	$\cdot 1946$	$4\cdot 11$	$5\cdot 358$
308	$-\cdot 7880$	$\cdot 6157$	$\cdot 1529$	$5\cdot 14$	$5\cdot 377$
310	$-\cdot 7660$	$\cdot 6428$	$\cdot 0716$	$10\cdot 69$	$5\cdot 410$

These values can be plotted, if the exact result is required, but it is near enough to take $\theta = 308°$.

$$\text{Then depth of water} = r\left(1 + \cos\frac{(360° - 308°)}{2}\right)$$

$$= r\,(1 + \cos 26°)$$

$$= 1{\cdot}8988\,r = {\cdot}949 \times \text{diam.} \quad \text{Ans.}$$

7. (M) *The efficiency of a square-threaded screw or a worm gear is given by the formula* $\eta = \dfrac{\tan a}{\tan(a+\phi)}$, *where* a *is the angle of the thread and* ϕ *is the angle of friction. For a given value of* ϕ *find what angle of thread will give a maximum efficiency.*

$$\eta = \frac{\tan a}{\tan(a+\phi)}.$$

The maximum value of η is given by $\dfrac{d\eta}{da} = 0$

$$\text{i.e.} \quad \frac{\tan(a+\phi)\dfrac{d\tan a}{da} - \tan a\dfrac{d\tan(a+\phi)}{da}}{\tan^2(a+\phi)} = 0$$

One possible solution of this is $\dfrac{1}{\tan^2(a+\phi)} = 0$

$$\text{i.e. } a + \phi = \frac{\pi}{2}$$

but this would make $\eta = 0$ and clearly does not give a maximum. Multiplying through by $\tan^2(a+\phi)$ we get

$$\tan(\alpha + \phi)\sec^2\alpha - \tan\alpha\sec^2(\alpha + \phi) = 0$$

i.e. $$\frac{\sin(\alpha + \phi)}{\cos(\alpha + \phi)\cos^2\alpha} - \frac{\sin\alpha}{\cos\alpha\cos^2(\alpha + \phi)} = 0.$$

Multiplying through by $\cos^2\alpha \,.\, \cos^2(\alpha + \phi)$,

$$\sin(\alpha + \phi)\cos(\alpha + \phi) - \sin\alpha\cos\alpha = 0$$

$$\text{i.e. } \tfrac{1}{2}\sin 2(\alpha + \phi) - \tfrac{1}{2}\sin 2\alpha = 0$$

$$\text{i.e. } \sin 2(\alpha + \phi) = \sin 2\alpha$$

$$\text{i.e. } 2(\alpha + \phi) = 180^\circ - 2\alpha$$

$$4\alpha = 180^\circ - 2\phi$$

$$\alpha = 45^\circ - \frac{\phi}{2}. \quad \text{Ans.}$$

8. (E) *The force exerted by a circular current of radius* a *on a small magnet whose axis coincides with the axis of the circle varies as* $\frac{x}{(a^2 + x^2)^{\frac{5}{2}}}$, *where* x *is the distance of the magnet from the plane of the circle. Prove that the force is a maximum when* $x = \frac{1}{2}a$.

Here $F = \frac{Kx}{(a^2 + x^2)^{\frac{5}{2}}}$, where K is a constant.

$\frac{dF}{dx} = 0$ will give the maximum value of F

$$\therefore \frac{(a^2 + x^2)^{\frac{5}{2}}\frac{dx}{dx} - x\frac{d\,(a^2 + x^2)^{\frac{5}{2}}}{dx}}{(a^2 + x^2)^5} = 0$$

$$(a^2 + x^2)^{\frac{5}{2}} . 1 - x \,.\, \tfrac{5}{2} \,.\, (a^2 + x^2)^{\frac{5}{2} - 1} \,.\, \frac{dx^2}{dx} = 0$$

$$(a^2 + x^2)^{\frac{5}{2}} - \frac{5x}{2}(a^2 + x^2)^{\frac{3}{2}} . 2x = 0$$

$$(a^2 + x^2)^{\frac{3}{2}}\{(a^2 + x^2) - 5x^2\} = 0$$

$$\therefore a^2 - 4x^2 = 0$$

$$\text{or } x = \pm\frac{a}{2}.$$

The positive and negative results represent positions on either side of the circle.

§ 38.—**Points of Inflexion in General.**—We have already mentioned points of inflexion in §§ 3, 13, and 36. They are points such as F (Fig. 28) where the direction of curvature is reversed, i.e. where the curve is momentarily straight. It will be noted that the slope $\frac{dy}{dx}$ diminishes (i.e. the tangent turns in a clockwise direction) as we approach F when it reaches a minimum value and then increases. Hence $\frac{d^2y}{dx^2} = 0$ at a point of inflexion, but $\frac{dy}{dx}$ does not vanish in general: the cases where it does vanish, i.e. when the point of inflexion has a horizontal tangent were mentioned in § 36.

Taking the example already used (p. 100)

$$y = 2x^3 - 3x^2 + 1,$$

we have

$$\frac{d^2y}{dx^2} = 12x - 6 = 0$$

$\therefore$ the point of inflexion occurs when

$$x = \tfrac{1}{2}.$$

Example of Point of Inflexion.—An engineering formula in which the point of inflexion is of interest is Rankine's formula for mild steel columns and struts which may be written

$$f_P = \frac{6}{1 + \dfrac{c^2}{6000}}$$

where c is the *buckling factor* (length divided by least radius of gyration for pin ends) (see p. 60, Ex. 2).

Expressing this in general terms we have

$$f_P = \frac{a}{1 + bc^2},$$

where a, b are constants.

$$\frac{df_P}{dc} = \frac{(1 + bc^2)\,0 - a\,(2bc)}{(1 + bc^2)^2}$$

$$= \frac{-\,2abc}{(1 + bc^2)^2}$$

$$\frac{d^2f_P}{dc^2} = 0 = -\,2ab\,\frac{d}{dc}\,\frac{c}{(1 + bc^2)^2}$$

$$= -\,2ab\left[\frac{1\,.\,(1 + bc^2)^2 - c\,.\,2\,(1 + bc^2)\,.\,2bc}{(1 + bc^2)^4}\right]$$

$$= -\,2ab\,\frac{1 + bc^2 - 4bc^2}{(1 + bc^2)^3} = -\,2ab\,\frac{1 - 3bc^2}{(1 + bc^2)^3}$$

Equating the numerator to zero,

$$1 - 3bc^2 = 0$$

$$\text{or } c = \pm\,1/\sqrt{3b}$$

The positive value is the only possible one

$$\therefore c = \frac{1}{\sqrt{\dfrac{3}{6000}}} = \sqrt{2000}$$

$$= 44.7 \text{ approx.} \quad \text{Ans.}$$

Compare this with the result obtained by plotting in Exercise II, 2.

Exercises 5b.

Further problems (mostly more difficult) on maxima and minima.

1. Find the values of x which will give respectively the maximum and minimum values of

$$2x^3 - 3x^2 - 36x + 8.$$

2. Find the maximum and minimum values of $\dfrac{(x - 1)^2}{(x + 1)^3}$.

3. Prove that $\frac{\log x}{x}$ has a maximum value when $x = e$.

4. The efficiency of a Pelton wheel is given by the formula $\eta = \frac{4\,v\,(\mathrm{U} - v)}{\mathrm{U}^2}$, where U is the velocity of the impinging water and v is the velocity of the vane. Find the maximum efficiency and the relative values of U and v at which it occurs.

5. If the strength of a beam of breadth b and depth d varies as bd^2, show that the strongest beam that can be cut out of a circular log of diameter D has a breadth of ·577 D.

6. In steaming against a tide, the resistance of the ship may be taken as proportional to the square of the velocity and the cost of fuel per hour is proportional to the product of the resistance and the velocity. Find the most economical velocity relatively to the tide for going a given distance.

7. In planning a bridge of several spans over a river, the cost of main girders for each span may be taken as proportional to the square of its span. Show that the most economical arrangement of equal spans will arise when the spans are such that the cost of main girders for each span is equal to the cost of each pier.

8. The bending moment at a certain point at distance x from one end of a beam of span l is equal to $\frac{\mathrm{W}x}{3} - \frac{\mathrm{W}x^3}{3l^2}$. Find the maximum bending moment.

9. The flow of steam through an orifice is proportional to $\alpha^{\frac{1}{\gamma}}\sqrt{1 - \alpha^{\left(\frac{\gamma-1}{\gamma}\right)}}$, where α is the ratio of pressures on the two sides and γ is a constant. Find the value of α to give a maximum flow.

10. Divide the number 20 into two parts such that their product is a maximum.

11. Find the least area of canvas that can be used to construct a conical tent of cubic capacity 800 cubic feet.

12. If a triangle has a given base and the sum of the other two sides is given, show that the area is greatest when these two sides are equal.

13. A rectangle of sides x, y has four square corners cut out and the sides turned up so as to form a rectangular box. Show that for the maximum volume the depth will be $\frac{1}{6}(x + y) - \sqrt{x^2 + y^2 - xy}$.

14. Find the height of a cylinder which is cut out of a sphere of radius R to give the maximum curved surface to the cylinder.

15. The efficiency of a certain hydraulic motor is given by

$$\eta = \frac{\dfrac{v(\mathrm{U} - v)}{g} - \dfrac{t}{2}\left(\dfrac{\mathrm{U}}{v} - \dfrac{v}{\mathrm{U}}\right)}{\dfrac{\mathrm{U}^2}{2g}},$$

where v is the variable. Prove that the maximum efficiency is attained when

$$2v^2\mathrm{U}^2 - 4v^3\mathrm{U} + gt\mathrm{U}^2 + gtv^2 = 0.$$

ANSWERS TO EXERCISES.

5A.

1. $26\frac{3}{4}$ at $x = 2\frac{1}{2}$. Maximum. **2.** 2 at $x = 0$. **3.** $-653\sqrt{3}$. **4.** 21·125. **5.** Half the area of the Δ. **7.** Maximum at $x = 1$; inflexion at $x = 2$. **9.** $2a^{\frac{1}{2}}b^{\frac{1}{2}}$. **10.** $\frac{\pi}{4}$.

5B.

1. Maximum when $x = -2$ and minimum when $x = +3$. **2.** $\frac{2}{27}$; 0. **4.** 1; $v = \frac{\mathrm{U}}{2}$. **6.** $1\frac{1}{2}$ times the velocity of the current. **7.** If P = cost of each pier, n = number of spans, l each span and L = total span and cost each span $= a \cdot \left(\frac{\mathrm{L}}{n}\right)^2$, there are

$(n-1)$ piers, therefore total cost $= (n-1)\mathrm{P} + na\,l^2$ $= \left(\frac{\mathrm{L}-l}{l}\right)\mathrm{P} + \mathrm{L}al$. Then differentiate with respect to l. **8.** ·128 Wl. **9.** $a = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$. **10.** 10 and 10. **11.** 361 square feet. **13.** Let z = height which will be the side of each square corner cut out, therefore dimensions of base of box will be $x-2z$, $y-2z$ and volume $z\,(x-2z)\,(y-2z)$. Differentiate with regard to z and equate to 0 thus getting a quadratic equation with the required solution. **14.** 1·414 R.

CHAPTER VI.

SYSTEMATIC INTEGRATION.

§ 39.—**Integrals.**—We have already discussed integrals from the graphical point of view in Chap. I, §§ 6, 7. We will now resume this discussion. Let us suppose y to be given in terms of x; suppose, for instance,

$$y = x^3.$$

We can then determine the slope of y,

$$\frac{dy}{dx} = 3x^2.$$

Conversely, if we started with

$$y_1 = 3x^2,$$

then x^3 would be the *indefinite integral* * of y, and we should write

$$† \int y dx = \int 3x^2 dx = x^3.$$

Graphically we should say that

$$y = 3x^2$$

being the derived curve of $y = x^3$, it follows that

$$y = x^3$$

is the sum curve of $y = 3x^2$.

Integration then is the inverse process to differ-

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* We call it "indefinite" because it is the general expression for the integral; in any given problem we have a particular or "definite" integral determined by the special conditions of the problem (see later).

† Read this *integral ydx.*

entiation. To *integrate any expression* means to answer the question: what other expression differentiated will give this one? Although graphically integration is simpler and more satisfactory than differentiation, analytically it is very much more complex and difficult. The processes of differentiation and integration are related to each other exactly in the same way as multiplication and division. We know that 72 ÷ 9 = 8 because we remember that 8 × 9 = 72. It will be noted that whenever we differentiate an expression, we learn an integral, e.g. by differentiating x^3 we learn the integral of $3x^2$; this is almost our only method of integrating; and in fact for many expressions, the integral cannot even be expressed at all in terms of the three catagories of laws dealt with in Chap. II. This is the case for example with

$$\int \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \text{ and } \int e^{-x^2} dx,$$

both of which arise in some problems.

On the other hand, the integration of a large number of simple expressions is easy enough, and this is so with the greater part of those occurring in a course of engineering.

In the first place we can make a table of standard integrals from our list of standard differentiations in § 32.

LIST OF STANDARD INTEGRALS.

	y	$\int y\,dx$ *
1.	x^n (except $n = -1$)	$\dfrac{x^{n+1}}{n+1}$
2.	$\dfrac{1}{x}$	$\log x$
3.	$\sin x$	$-\cos x$
4.	$\cos x$	$\sin x$
5.	$\tan x$	$\log \sec x$
6.	$\cot x$	$\log \sin x$
7.	$\sec x$	$\log \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$ $= \log\left(\dfrac{1+\sin x}{\cos x}\right)$ $= \log\left(\dfrac{\cos x}{1-\sin x}\right)$
8.	$\operatorname{cosec} x$	$\log \tan \dfrac{x}{2}$
9.	$\sec^2 x$	$\tan x$
10.	$-\operatorname{cosec}^2 x$	$\cot x$
11.	$1/\sqrt{1-x^2}$	$\sin^{-1} x$
12.	$1/(1+x^2)$	$\tan^{-1} x$
13.	$1/(1-x^2)$	$\frac{1}{2} \log \dfrac{1+x}{1-x}$
14.	e^x	e^x
15.	$\log x$	$x \log x - x.$

Some of these do not occur in the table of differentiations, but all are important, and the student should endeavour to learn them by heart. *All should be verified by differentiation.*

Arbitrary Constant.—The differential coefficient of a constant is zero. Hence if any arbitrary constant be added to an indefinite integral, it will still do equally well. Hence *to all the indefinite integrals* which we obtain *an arbitrary constant C must be supposed to be added.* Thus the general form for

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* $\int dx$ means $\int 1\,dx$ and is x.

$$\int x^n \, dx \text{ is } \frac{x^{n+1}}{n+1} + C$$

This point is well brought out by a diagram (Fig. 51). The effect of adding a constant C to an expression is to raise the graph through a height C, as in the curves shown. The two curves, however, have everywhere the same slope and therefore the same slope curve, and the two expressions of which these are the graphs will be integrals of the same expression.

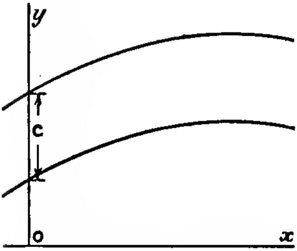


FIG. 51.—Effect of adding a constant to an expression.

The student can now determine a number of integrals for himself. Thus to integrate $(x^2 + 1)$, we see at once that this will be obtained by differentiating $(\frac{1}{3}x^3 + x + C)$.

EXERCISES 6A.

Evaluate the following indefinite integrals :—

1. $\int (x+1)\, dx$. 2. $\int (bx+a)\, dx$. 3. $\int (ax^2+bx+c)\, dx$.

4. $\int \frac{dx}{\sqrt{x}}$. 5. $\int \frac{3dx}{x^{1 \cdot 41}}$. 6. $\int \left(\frac{x+2}{x}\right) dx$.

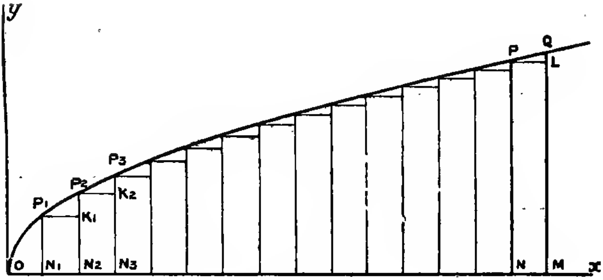
7. $\int (1-\cos 2x)\, dx$. 8. $\int \tan^2 x dx = \int (\sec^2 x - 1) dx$.

9. $\int(A \sin mx + B \cos mx)dx$. 10. $\int\frac{dx}{1 + a^2x^2}$.

§ 40.—**Definite Integrals.**—There are two aspects of integration: it first appears as the inverse process to differentiation, and then as a summation. We come now to the second aspect, and we shall afterwards establish the connexion between the two.

Consider the curve (a parabola)

$$y = \sqrt{x} = x^{\frac{1}{2}}$$



$y = x^{\frac{1}{2}} = \sqrt{x}$

FIG. 52.

Let NP be any ordinate, where ON = x, and let us try to find the area between the curve OP and the abscissa ON.

Divide the abscissa ON into a large number n of equal parts $ON_1 = N_1N_2 = N_2N_3 = \ldots = \frac{x}{n}$. Draw the corresponding ordinates N_1P_1, N_2P_2, N_3P_3 . . . and draw P_1K_1, P_2K_2, . . . parallel to the x-axis.

Then the sum of the rectangles N_1K_1, N_2K_2, . . . is less than the area under the curve but approaches it in value, the defect being the sum of the triangular pieces ON_1P_1, $P_1K_1P_2$, $P_2K_2P_3$. . . . If the number n of slices into which we cut the area is increased the approximation will become closer, for the triangular pieces form a vanishing area bordering the curve.

So that when n is sufficiently great, the error is below our limits of accuracy, and the sum of the rectangles becomes the area under the curve.

The quantities $\frac{x}{n}$ may be considered as increments in x and written δx, while 0, N_1P_1, N_2P_2 . . . being successive ordinates may be written $y_o = 0, y_1, y_2 . . . .$

Thus the area under the curve is ultimately equal to

$$y_1\delta x + y_2\delta x + y_3\delta x + \ldots$$

Using the symbol S to mean "*the sum of all expressions like,*" the area

$$= \underset{x=o}{\overset{x=x}{\mathrm{S}}} y\delta x,$$

the extreme values of the abscissa being indicated below and above the S. When n is sufficiently great for the limits of accuracy to be surpassed (i.e. the error of the approximation is less than the errors of calculation, measurement, etc.), we lengthen the S:

$$\mathrm{A} = \text{Area} = \int_o^x y\,dx,$$

"the integral of y from o to x".

We can suppose that we take the area for different values of the variable x, and then A would have an expression in terms of x. Let us consider what expression this is.

Let us consider a small change in A

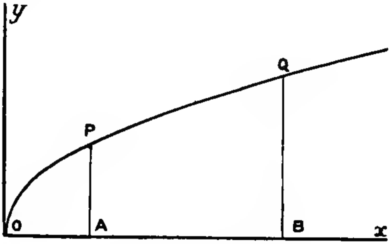
$$\begin{aligned}\delta A &= \text{area NMQP} \\ &= \text{rectangle NMLP (approx.)} \\ &= \text{NP} \times \text{NM} \\ &= y\delta x.\end{aligned}$$

So that

$$\begin{aligned} y &= \frac{\delta A}{\delta x} \text{ (approx.)} \\ &= \frac{dA}{dx} \text{ (exactly).}\end{aligned}$$

So that A is an expression which when differentiated gives y (or $x^{\frac{1}{2}}$), i.e. A *is an integral of* $y = x^{\frac{1}{2}}$.

Referring to our table of integrals (No. 1, p. 123) we see that



$y = x^{\frac{1}{2}} = \sqrt[2]{x}$.

FIG. 53.

$$A = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = \tfrac{2}{3}x^{\frac{3}{2}} + C.$$

But we do not yet know what value of C must be taken. To find what value C has *in this particular case,* note that A = 0 when $x = 0$, and substitute

$$0 = \tfrac{2}{3}\,0^{\frac{3}{2}} + C, \text{ i.e. } C = 0.$$

Thus

$A = \frac{2}{3}x^{\frac{3}{2}} = \frac{2}{3}xy$ i.e. $\frac{2}{3}$ × rectangle ON × NP

Suppose we wished to find the area of a different section say between $x = a$ and $x = b$

Then area up to A $= \frac{2}{3}a^{\frac{3}{2}} + C$ (retaining C for the moment) and area up to B $= \frac{2}{3}b^{\frac{3}{2}} + C$. Hence

$$\text{area ABQP} = \int_a^b y\,dx = \left(\tfrac{2}{3}b^{\frac{3}{2}} + C\right) - \left(\tfrac{2}{3}a^{\frac{3}{2}} + C\right)$$
$$= \tfrac{2}{3}b^{\frac{3}{2}} - \tfrac{2}{3}a^{\frac{3}{2}}.$$

So that C disappears and any indefinite integral could have been taken. This is why we have not put in the arbitrary constants in the list of standard integrals.

These integrals with upper and lower limits are called *definite integrals*, and *to find a definite integral, first find the indefinite integral, then substitute in it the upper and lower limits, and then subtract the latter result from the former.*

In practice the definite integral expresses a result under definite specified conditions, while the indefinite integral expresses the same result when the conditions are not specified. The former, for example, might be the distance covered by a moving body starting from a given point, the latter the distance covered from any starting-point.

The following are examples: *note the square bracket notation*:—

$$\int_a^b x^n\,dx = \left[\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

$$\int_a^b e^x\,dx = \left[e^x\right]_a^b = e^b - e^a$$

$$\int_0^{\frac{\pi}{2}} \sin x\,dx = \Big[-\cos x\Big]_0^{\frac{\pi}{2}} = \left(-\cos\frac{\pi}{2}\right) - (-\cos 0)$$
$$= 0 - (-1) = 1$$

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 - \tan^{-1}0$$

$$= \frac{\pi}{4} - 0$$

$$= \frac{\pi}{4} = \cdot785.$$

Note.—We have supposed that the divisions δx were all equal: the value of the integral is not altered if they are not all equal, provided only that they all become indefinitely small. The proof of this is too difficult to insert here.

Exercises 6b.

Evaluate the following definite integrals:—

1. $\int_1^4 x^2 dx.$ 2. $\int_0^1 dx.$ 3. $\int_{10}^{20} 2\ (4 + x)\ dx$

4. $\int_{-1}^{+1}(x - x^3)\, dx.$ 5. $\int_1^4 \frac{dx}{2\sqrt{x}}.$ 6. $\int_{-1\cdot5}^{6}(2x + 3)^2 dx.$

7. $\int_0^1 \frac{dx}{\sqrt{1 - x^2}}.$ 8. $\int_0^1 \frac{dx}{1 + x^2}.$ 9. $\int_0^{\frac{\pi}{2}} \sin^2 x\, dx.$

10. $\int_0^{\frac{\pi}{2}} \sec^2 x\, dx.$

§ 41. **Applications of Integrals.**—Applications of both indefinite and definite integrals are very varied and numerous. We shall discuss several in some detail in the next chapter. But we will give a few instances here.

Consider again the nature of an integral. We take the values of a dependent quantity y for a whole sequence of very near values of x, we multiply each by the corresponding increment and we add. Any quantity z which is obtained from another y in this way is an integral, and the area of a curve is only a special case.

1°. Mass of a Rod of varying Density.—Let the length of the rod be 1 ft., and let its density ρ, i.e. its mass per unit length, at any point be proportional to the distance from one end, and be 3 lb. per foot at the other end.

The density at distance x from the front end $= \frac{x}{1} \times 3 = 3x$, so that $\rho = 3x$.

Suppose the rod to be divided into a number of equal parts of which δx is one.

Then the mass of this part $= \rho\delta x$ (approx.), and the whole mass is

$$S\rho\delta x \quad \text{(approx.)}$$

or

$$\int_o^1 \rho\delta x \text{ (exactly).}$$

$$\text{So mass } = \int_o^1 \rho dx = \int^1 3\, x\, dx$$

$$= \left[3\,\frac{x^2}{2}\right]_o^1 = \frac{3}{2} \text{ lb.}$$

2°. TO FIND THE VELOCITY AT ANY INSTANT AND THE DISTANCE FALLEN THROUGH BY A BODY FALLING FREELY UNDER GRAVITY.—The body is dropped from a given point, and moves with a uniform acceleration $= 32$ feet per second per second.

If x is the distance it has fallen in time t, then the velocity is $\frac{dx}{dt}$ and the acceleration

$$\frac{d^2x}{dt^2} = 32.$$

Now $\frac{dx}{dt}$ is the quantity which, when differentiated, gives 32. So

$$\frac{dx}{dt} = \int_o^t 32\, dt = [32t]_o^t$$

$$= 32t.$$

Similarly x is the integral of the velocity

$$x = \int_0^t 32t\,dt$$

$$= \left[32 \cdot \frac{t^2}{2}\right]_0^t = 32\,\frac{t^2}{2}$$

$$= 16\,t^2.$$

§ 42. **Reduction to Standard Form.**—So far we know the fifteen integrals given in the standard list of § 39, and such others as we may arrive at by differentiation. We can obtain the other integrals by reducing them to standard ones by various means.

SIMPLE CASES.

1°. MULTIPLYING BY A CONSTANT.—Looking at an integral from the summation point of view we see that the integral of *a times y = a times the integral of y*, i.e. $\int aydx = a\int ydx$.

For example

$$\int a \sin x\,dx = -a \cos x.$$

2°. MULTIPLYING THE ARGUMENT * BY A CONSTANT.—We will find the integral of sin bx,

$$\int \sin bx\,dx.$$

It is clear that an increment in bx

$$\delta bx = b\delta x.$$

If then in the sum we replace each increment δx by δbx, we shall be multiplying each element and therefore the whole by b. Hence

$$\int \sin bx\,dx = \frac{1}{b}\int \sin bx\,d(bx)$$

$$= \frac{1}{b}(-\cos bx)$$

$$= -\frac{1}{b}\cos bx.$$

---

* The *argument* of sin x is x, of sin bx is bx, and in general of $f(x)$ is x.

3°. ADDITION OF CONSTANT TO ARGUMENT.—Consider the integral of $\sin(x+c)$. An increment in $(x+c)$ is the same as an increment in x, viz. δx,

$$\text{i.e. } \delta(x+c) = \delta x.$$

Hence

$$\int \sin(x+c)\,dx = \int \sin(x+c)\,d(x+c)$$
$$= -\cos(x+c).$$

Combining 1°, 2° *and* 3°

$$\int a \sin(bx+c)\,dx = a \,.\, \frac{1}{b}\int \sin(bx+c)\,d(bx+c)$$
$$= -\frac{a}{b}\cos(bx+c).$$

To take another example

$$\int a^x dx = \int (e^{\log_e a})^x dx$$
$$= \int e^{x\log_e a} dx$$
$$= \frac{1}{\log_e a}\int e^{x\log_e a} d(x\log_e a)$$
$$= \frac{1}{\log_e a}\, e^{x\log_e a}$$
$$= \frac{a^x}{\log_e a}.$$

And another

$$\int \log_{10}x\,dx = \int(\log_{10}e \times \log_e x)\,dx$$
$$= \log_{10}e\int \log_e x\,dx$$
because $\log_{10}e$ is a constant
$$= \log_{10}e\,(x\log_e x - x)$$
$$= x\log_{10}e\,\log_e x - x\log_{10}e$$
$$= x\log_{10}x - x\log_{10}e.$$

4°. THE INTEGRAL OF A SUM is the sum of the corresponding integrals. Thus

$$\int(x^2+3x+2)\,dx = \int x^2dx + 3\int x\,dx + 2\int dx$$
$$= \frac{x^3}{3} + 3\frac{x^2}{2} + 2x.$$

5°. TRANSFORMATION OF "INTEGRAND".—Consider the following cases :—

$$\begin{aligned}(1)\ \int(x+1)^2\,dx &= \int(x^2+2x+1)\,dx\\ &= \int x^2dx + 2\int xdx + \int dx\\ &= \frac{x^3}{3} + 2\,.\,\frac{x^2}{2} + x\\ &= \tfrac{1}{3}\,(x^3+3x^2+3x).\end{aligned}$$

This may also be done as follows by 3° :—

$$\begin{aligned}\int(x+1)^2\,dx &= \int(x+1)^2\,d\,(x+1)\\ &= \tfrac{1}{3}\,(x+1)^3\\ &= \tfrac{1}{3}\,(x^3+3x^2+3x+1).\end{aligned}$$

Observe that these two integrals of the same expression differ by a constant, viz. $\frac{1}{3}$. This does not matter as they are *indefinite.*

$$\begin{aligned}(2)\ \int\sin^2 x dx &= \int\tfrac{1}{2}\,(1-\cos 2x)\,dx\\ &= \tfrac{1}{2}\int dx - \tfrac{1}{2}\int\cos 2xdx\\ &= \tfrac{1}{2}\int dx - \tfrac{1}{4}\int\cos 2x\,d\,(2x)\\ &= \tfrac{1}{2}\,x - \tfrac{1}{4}\sin 2x\\ &= \tfrac{1}{2}\,(x - \cos x\sin x).\end{aligned}$$

6°. PARTIAL FRACTIONS. — To integrate $\dfrac{1}{1-x^2}$, we may write

$$\frac{1}{1-x^2} = \tfrac{1}{2}\,\frac{1}{1-x} + \tfrac{1}{2}\,\frac{1}{1+x} \quad . \quad . \quad . \quad (1)$$

Then

$$\begin{aligned}\int\frac{dx}{1-x^2} &= \int\left[\tfrac{1}{2}\,\frac{1}{1-x} + \tfrac{1}{2}\,\frac{1}{1+x}\right]dx\\ &= \tfrac{1}{2}\int\frac{dx}{1-x} + \tfrac{1}{2}\int\frac{dx}{1+x}\\ &= -\tfrac{1}{2}\int\frac{dx}{x-1} + \tfrac{1}{2}\int\frac{dx}{x+1}\end{aligned}$$

$$= -\tfrac{1}{2}\log\,(x-1) + \tfrac{1}{2}\log\,(x+1) = \log\sqrt{\frac{x+1}{x-1}}.$$

To effect the transformation (1), write

$$\frac{1}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x}$$

where A and B are "undetermined coefficients".

Multiply up by $1 - x^2$, then

$$1 = A(1 + x) + B(1 - x) \quad . \quad . \quad . \qquad (2)$$

Putting $x = 1$, $1 = 2A$, and $A = \frac{1}{2}$.

Putting $x = -1$, $1 = 2B$, and $B = \frac{1}{2}$.

Or we may "equate the coefficients" on the two sides. Thus (2) is

$$1 = A + B + (A - B)x.$$

$$\text{Hence } 1 = A + B,\ O = A - B,$$

whence, as before, $A = B = \frac{1}{2}$.

§ 43.—**Transformation of Independent Variable.** —This is the most useful method for reducing to standard form, and depends upon the "function of a function" formula. If y depends upon z and z depends upon x, we have (§ 27, 4°)

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Suppose we have an expression in z which we wish to integrate, say,

$$y_1 = \frac{dy}{dz}.$$

Then our integral is

$$y = \int y_1 dz.$$

But we may not be able to integrate this and so find y. We may then use our formula which gives us the slope of y with respect to another variable x; for we have

$$y = \int \frac{dy}{dx}\, dx = \int \frac{dy}{dz} \cdot \frac{dz}{dx}\, dx,$$

that is to say an integration in terms of x instead of z; and this new integral we may be able to find.

Suppose we wish to integrate $\dfrac{1}{\sqrt{1-z^2}}$ (one of our standard integrals).

We want then

$$\int\frac{dz}{\sqrt{1-z^2}}.$$

Put $z = \sin x$. The integral

$$\begin{aligned}\int\frac{dz}{\sqrt{1-z^2}} &= \int\frac{1}{\sqrt{1-z^2}}\frac{dz}{dx}\,dx\\ &= \int\frac{1}{\sqrt{1-z^2}}\cos x\,dx\\ &= \int\frac{1}{\sqrt{1-\sin^2 x}}\cos x\,dx\\ &= \int\frac{\cos x}{\cos x}\,dx = \int dx = x.\end{aligned}$$

(Note that when integrating with respect to x we must put everything in terms of x.)

But we want our integral in terms of z: we have then

$$\text{integral} = x = \sin^{-1} z.$$

The general formula may be written

$$\int y_1 dz = \int y_1\frac{dz}{dx}\,dx,$$

and therefore it comes to replacing

$$dz \text{ by } \frac{dz}{dx}\,dx.$$

This is in reality only the formula for a small variation, $\delta z = \dfrac{dz}{dx}\delta x$.

It will be asked how we arrive at the particular transformation

$$z = \sin x.$$

How do we know this will enable us to perform the integration? We do not know. We can only make trials of likely transformations until we arrive at the right one. But a little experience will soon put us upon the right track. It may be remarked that a radical like

$$\sqrt{1 - z^2}$$

may be reduced by putting $z = \sin x$, and similarly for

$$\sqrt{z^2 + 1} \text{ we put } z = \tan x$$
$$\sqrt{z^2 - 1} \text{ we put } z = \sec x.$$

The formula may often be applied from the opposite point of view, i.e. we can regard our original integral as having the form

$$\int y_1 \frac{dz}{dx}\, dx\,;$$

and put it into the form

$$\int y_1 dz.$$

Examples:—

(1) $$\int \frac{1}{x} \log x\, dx:$$

note that $$\frac{1}{x} = \frac{d \log x}{dx}.$$

So we have

$$\int \log x \frac{1}{x}\, dx = \int \log x \frac{d \log x}{dx} dx$$
$$= \int \log x \;\, d \log x = \tfrac{1}{2} (\log x)^2$$

(by standard integral No. 1).

(2) $\int x \sin x^2\, dx = \frac{1}{2}\int \sin x^2 \,.\, 2x dx$

$(2x dx = dx^2)$

$= \frac{1}{2}\int \sin x^2 dx^2$

$= - \frac{1}{2} \cos x^2$

(by standard integral No. 3).

$$(3)\int \tan x dx = \int \frac{\sin x dx}{\cos x}$$

$$(-\sin x dx = d\cos x)$$

$$= -\int \frac{d\cos x}{\cos x}$$

$$= -\log\cos x$$

$$= \log\frac{1}{\cos x} = \log\sec x$$

(by standard integral No. 2 : this is No. 5).

§ 44. **Integration by Parts.**—This method depends upon the product formula in differentiation (§ 27, 2°).

$$\frac{dy_1y_2}{dx} = y_2\frac{dy_1}{dx} + y_1\frac{dy_2}{dx}.$$

Rearranging

$$y_1\frac{dy_2}{dx} = \frac{dy_1y_2}{dx} - y_2\frac{dy_1}{dx}.$$

Now integrate both sides of this equation.

$$\int y_1\frac{dy_2}{dx}dx = y_1y_2 - \int y_2\frac{dy_1}{dx}dx$$

[which may also be written (see last §)

$$\int y_1dy_2 = y_1y_2 - \int y_2dy_1].$$

To take an example consider

$$\int x\cos x\, dx;$$

cos x is the differential coefficient of sin x. So we have

$$\int x\cos x\, dx = \int x\frac{d\sin x}{dx}dx$$

$$= x\sin x - \int \sin x\frac{dx}{dx}dx$$

$$= x\sin x - \int \sin x\, dx$$

$$= x\sin x + \cos x.$$

The original integral is put in terms of another which can be integrated. This method is useful when we have x (or x^2, x^3, etc.) multiplying an integrable expression. The formula may be applied several times. Thus to integrate

$$\begin{aligned}\int x^2 \cos x\,dx &= \int x^2\,d\sin x\\ &= x^2 \sin x - \int \sin x\,dx^2\\ &= x^2 \sin x - \int \sin x \,.\, 2\,x\,dx\\ &= x^2 \sin x - 2\int x \sin x\,dx\\ &= x^2 \sin x - 2\,(-\,x\cos x + \sin x)\\ &= x^2 \sin x + 2x\cos x - 2\sin x.\end{aligned}$$

Another example is

$$\int \log x\,dx:$$

here we take

$$y_1 = \log x,\ y_2 = x$$

$$\begin{aligned}\text{then } \int \log x\,dx &= x\log x - \int x \,.\, d\log x\\ &= x\log x - \int x \,.\, \frac{1}{x}\,dx\\ &= x\log x - \int dx\\ &= x\log x - x.\end{aligned}$$

EXERCISES 6c.

Evaluate the following integrals, hints being given in many cases as to the method to adopt :—

1. $\int\frac{dx}{2x\quad 4}$. 2. $\int\frac{dx}{\sqrt{2x+5}}$. 3. $\int\frac{x}{x^2+1}dx$, substituting $z = x^2 + 1$. 4. $\int\frac{dx}{x^2 - a^2}$, working by partial fractions. 5. $\int\frac{5x-7}{(x+1)\,(2x-1)}\,dx$, working by partial fractions. 6. $\int\frac{dx}{(x^2+r^2)^{\frac{3}{2}}}$, putting $x = r\tan\theta$.

7. $\int^b\frac{x\,dx}{r^2+x^2}$. 8. $\int\frac{dx}{2-x-x^2}$, by partial frac-

tions. 9. $\int \frac{x+1}{(x-1)^2}$. 10. $\int \frac{dx}{x(1+x^2)}$ putting $z = x^2$. 11. $\int \frac{xdx}{x^4 - 1}$ putting $z = x^2$. 12. $\int \frac{xd}{\sqrt{r^4 - x^4}}$. 13. $\int \sin^2 xdx$. 14. $\int \cos^2 (rx + b)$. 15. $\int \sin^2 x \cos xdx$. 16. $\int_0^\pi \cos^2 xdx$. 17. $\int x \cos xdx$, by parts. 18. $\int x \sin x \cos xdx$, by parts. 19. $\int xe^{\frac{x}{r}} dx$, by parts. 20. $\int x \sec^2 xdx$. 21. $\int \frac{dx}{(1+x^2)^2}$, putting $x = \tan \theta$. 22. $\int \frac{xdx}{\sqrt{x+1}}$, putting $x = z^2$. 23. $\int \frac{x^2dx}{1-x^4}$, by partial fractions. 24. $\int_0^1 \frac{xdx}{\sqrt{1-x^2}}$. 25. $\int_0^1 \frac{xdx}{\sqrt{1+x^2}}$. 26. $\int_1^2 \frac{dx}{\sqrt{x^2-1}}$. 27. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\cos\theta}$. 28. $\int \sqrt{\left(\frac{1-x}{1+x}\right)}dx$, by putting $x = \cos\theta$ and simplifying. 29. $\int \frac{dx}{1-4x+4x^2}$, putting $z = (2x-1)$ 30. $\int \tan^2 xdx$.

ANSWERS TO EXERCISES.

Exercises 6a.

1. $\frac{x^2}{2} + x$. 2. $\frac{bx^2}{2} + ax$. 3. $\frac{ax^3}{3} + \frac{bx^2}{2} + cx$. 4. $2\sqrt{x}$. 5. $\frac{-3}{\cdot 41\, x^{\cdot 41}}$. 6. $x + 2 \log x$. 7. $x - \frac{\sin 2x}{2}$. 8. $\tan x - x$. 9. $mA \cos mx - mB \sin mx$. 10. $\frac{1}{a} \tan \frac{x}{a}$.

Exercises 6b.

1. 21. 2. 1. 3. 380. 4. 0. 5. 1. 6. $\frac{5}{11}$. 7. $\frac{\pi}{2}$. 8. $\frac{\pi}{4}$. 9. $\frac{\pi}{4}$. 10. infinity.

EXERCISES 6c.

1. $\frac{1}{2}\log(2x-4)$. 2. $\sqrt{(2x+5)}$
3. $\frac{1}{2}\log(x^2+1)$. 4. $\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$. 5. $4\log(x+1)$
$+\frac{3}{2}\log(2x-1)$. 6. $\frac{x}{r^2\sqrt{(x^2+r^2)}}$. 7. $\frac{1}{2}\log\frac{r^2+b^2}{r^2}$.
8. $\frac{1}{3}\log\frac{x+2}{1-x}$. 9. $\log(x-1)-\frac{2}{x-1}$.
10. $\log\frac{x}{\sqrt{(1+x^2)}}$. 11. $\frac{1}{4}\log\frac{x^2-1}{x^2+1}$. 12. $\frac{1}{2}\sin^{-1}\frac{x^2}{r^2}$.
13. $\frac{1}{2}x-\frac{1}{4}\sin 2x$. 14. $\frac{x}{2}+\frac{1}{4n}\sin(nx+b)$.
15. $\frac{\sin^3 x}{3}$. 16. $\frac{\pi}{2}$. 17. $\sin x - x\cos x$.
18. $\frac{\sin 2x}{8}-\frac{x\cos 2x}{4}$. 19. $a(x-a)e^{\frac{a}{x}}$.
20. $x\tan x+\log\cos x$. 21. $\frac{1}{2}\tan^{-1}x+\frac{1}{2}\frac{x}{(1+x^2)}$.
22. $\frac{2}{3}x^{\frac{3}{2}}-x+2x^{\frac{1}{2}}-2\log(x^{\frac{1}{2}}-1)$. 23. $\frac{1}{4}\log\frac{1+x}{1-x}$
$-\frac{1}{2}\tan^{-1}x$. 24. 1. 25. $\sqrt{2}-1$.
26. $\log(2+\sqrt{3})$. 27. 1. 28. $\sin^{-1}x+\sqrt{(1-x^2)}$.
29. $-\frac{1}{2(2x-1)}$. 30. $\tan x - x$.

CHAPTER VII.

APPLICATIONS AND DEVELOPMENTS OF INTEGRATION.

§ 45.—We will now consider some of the simpler problems in engineering work in which integration arises. We wish here again to emphasize the fact that all integration is a matter of practice and experience, and in some problems we may get expressions to integrate that we cannot work out without very great trouble. When such an integration occurs we can always get a result which will be approximately correct by plotting against the variable the expression that we wish to integrate, and finding the

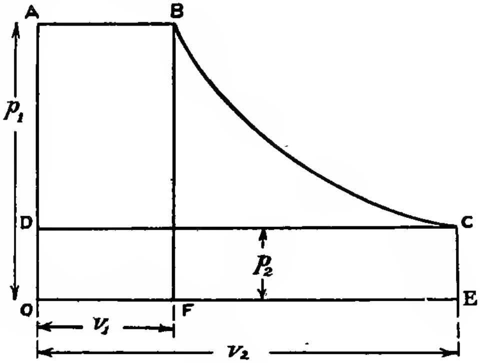


Fig. 54.

area of the resulting figure by planimeter or otherwise (Chapter I).

§ 46. **Work done in Engine Cylinder.**—Suppose that we have a steam or other heat engine in which gas or vapour is admitted at constant pressure p_1 up to a point B where it is cut off (Fig. 54). It then expands to a point C, and is then exhausted at constant pressure p_2.

Taking unit area of piston, the pressure represents the force on the piston and the volume represents the position in the stroke from the beginning. We have shown already (p. 19) that the work done is equal to the area of the diagram of force plotted against distance.

$$\begin{aligned}\therefore \text{work done} &= \text{area ABCD}\\ &= \text{area (ABFO + FBCE} - \text{ODCE)}\\ &= p_1v_1 + \int_{v_1}^{v_2} pdv - p_2v_2.\end{aligned}$$

Now in most gases the expansion can be expressed by the law $pv^n = \text{constant} = k$.

$$\begin{aligned}\therefore \int pdv &= \int \frac{kdv}{v_n} = k\int \frac{dv}{v^n}\\ &= k\int v^{-n}dv = \frac{kv^{-n+1}}{-n+1}\\ &= \frac{kv}{(1-n)\,v^n} = \frac{pv}{(1-n)}.\end{aligned}$$

$$\therefore \int_{v_1}^{v_2} pdv = \left[\frac{p_2v_2 - p_1v_1}{(1-n)}\right] = \frac{p_1v_1 - p_2v_2}{(n-1)}$$

ADIABATIC EXPANSION.—In this case n, in the equation $pv^n = k$ is γ, the ratio between the specific heats at constant pressure and volume respectively.

$$\therefore \int_{v_1}^{v_2} pdv = \frac{p_1v_1 - p_2v_2}{(\gamma - 1)}.$$

$$\therefore \text{ work done } = \text{area ABCD}$$

$$= p_1v_1 - p_2v_2 + \frac{p_1v_1 - p_2v_2}{\gamma - 1}$$

$$= (p_1v_1 - p_2v_2)\left(1 + \frac{1}{\gamma - 1}\right)$$

$$= \frac{\gamma}{\gamma - 1}(p_1v_1 - p_2v_2)$$

Isothermal Expansion.—In this case n is 1, i.e. $pv = k$, $\therefore p_1v_1 = p_2v_2$.

Our expression $\frac{p_1v_1 - p_2v_2}{(n - 1)}$ then becomes $\frac{0}{0}$ which is *indeterminate.*

We must therefore go back to the integral which gives

$$\int pdv = k\int\frac{dv}{v} = k \log_e v.$$

$$\therefore \int_{v_1}^{v_2} pdv = k\,(\log_e v_2 - \log_e v_1) = k \log_e \frac{v_2}{v_1}\,.\text{*}$$

The quantity $\frac{v_2}{v_1}$ is commonly called r, the ratio of expansion.

$$\therefore \int_{v_1}^{v_2} pdv = k \log_e r = p_1v_1 \log_e r.$$

$$\therefore \text{ work done } = \text{area ABCD}$$

$$= p_1v_1 + p_1v_1 \log_e r - p_2v_2$$

$$= p_1v_1 \log_e r \text{ (since } p_1v_1 = p_2v_2).$$

§ 47. **Mean Values.**—Let the diagram represent a varying quantity y. *Construct a rectangle OBTS having the same area as that under the curve. Then*

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* Because the logarithm of a quotient is log. dividend − log. divisor.

OS, the height of the rectangle, is the mean value of y between the limits O and B.

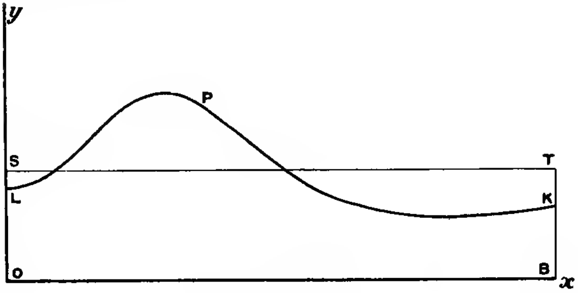

FIG. 55.

Let OB $= a$; then the area OBKPL $= \int_o^a y\,dx$.

Hence

$$\textit{the mean value of } y = \frac{1}{a}\int_o^a y\,dx.$$

If $\bar{y}$ = the mean value, then $a\bar{y} = \int_o^a y\,dx$.

Examples :—

1°. The mean value of $\sin x$ between $x = 0$ and $x = \pi$

$$= \frac{1}{\pi}\int_o^\pi \sin x\,dx = \frac{1}{\pi}\Big[-\cos x\Big]_o^\pi$$

$$= \frac{1}{\pi}\Big(-(-1)-(-1)\Big)$$

$$= \frac{2}{\pi} = \cdot 637.$$

2°. In the last article (§ 47) the mean pressure of the fluid between A and C will be $\dfrac{p_1v_1 - p_2v_2}{v_2\,(n-1)}$ and between B and C will be $\frac{p_1v_1 - p_2v_2}{(v_2 - v_1)(n - 1)}$.

§ 48. **Centres of Gravity.**—First Moments.—Taking two co-ordinate axes OX, OY in a plane, suppose that particles of matter $m_1, m_2, m_3, \ldots$ are situated at the points $P_1, P_2, P_3 \ldots$

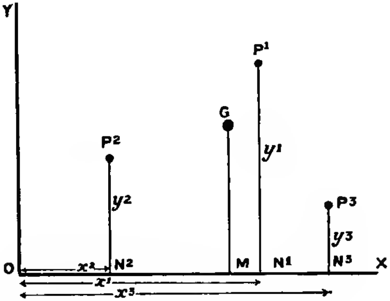

Fig. 56.

Then the first moment of the particle m_1 about the axis OX is

$$m_1y_1,$$

and the first moment of the system $m_1, m_2, m_3, \ldots$ is the sum of the first moments of the particles taken separately,

first moment of system about OX

$$= m_1y_1 + m_2y_2 + m_3y_3 + \ldots$$

Similarly for the first moment about any other line, for example about OY we have

$$m_1x_1 + m_2x_2 + m_3x_3 + \ldots$$

Centre of Gravity.—Suppose now that the whole mass

$$M = m_1 + m_2 + m_3 + \ldots$$

is concentrated at a point G, such that the first moments of M at G about OX and about OY are equal respectively to the first moments of the system about OX and about OY. If the co-ordinates of G are $(\bar{x}, \bar{y})$, then

$$M\bar{x} = m_1x_1 + m_2x_2 + m_3x_3 + \dots \text{ (about OY)}$$

and $$M\bar{y} = m_1y_1 + m_2y_2 + m_3y_3 + \dots \text{ (about OX).}$$

Then it can be proved that the first moment of M *at* G *and the first moment of the system, about any other axis are also equal.* So that as far as first moments are concerned, *the whole mass may be supposed concentrated at its centre of gravity.*

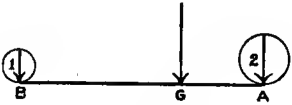

FIG. 57.

In particular the moment of the system about any axis through G is zero.

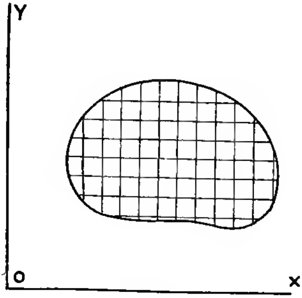

FIG. 58.

[Example: the centre of gravity of a mass of 2 lb. at A and a mass of 1 lb. at B is on AB and divides it so that 2AG = GB.]

§ 49. **Continuous Bodies.**—Continuous bodies may be supposed to be divided up into a large number of small parts as in Fig. 58, each of which may be treated as a particle. If we call each part δm (an "element of mass"), then the co-ordinates of the centre of gravity may be determined as before by the equations

$$M\bar{x} = x_1\delta m + x_2\delta m + x_3\delta m + \ldots$$
$$M\bar{y} = y_1\delta m + y_2\delta m + y_3\delta m + \ldots$$

approximately, or by

$$M\bar{x} = \int x dm, \; M\bar{y} = \int y dm.$$

In some elementary cases the position of G may be inferred from symmetry. Thus for uniform plates in the shapes of a square, a parallelogram, or a circle, it is clear that the centre of the figure is also the centre of gravity.

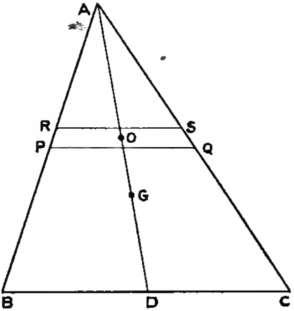

FIG. 59.

Again take a uniform triangular plate ABC; divide it up into thin rods such as PQSR parallel to the base BC. Then the median AD bisects each rod, and hence passes through the centre of gravity of each rod. Hence the triangle may be replaced by a number of particles placed along AD, and G must lie in AD. Similarly G must lie in each of the two other medians and must therefore be their point of intersection. And hence G is a point on DA, one-third of the way along it.

CENTROIDS.—In dealing with areas we often want to find the *centroid*, i.e. the point in the area which would be the centre of gravity of a uniform plate of the same shape as the area. In this case we proceed exactly as before, replacing mass by area in the expressions. In this case the formulæ become

$$A\bar{x} = \int x\,da, \quad A\bar{y} = \int y\,da$$

where A = total area of body

da = element of area.

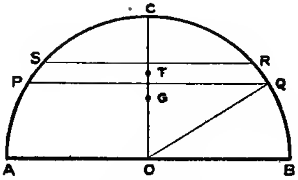

FIG. 60.

§ 50. **Examples.**—We will now take a few cases of centres of gravity and centroids in which integration is useful.

1°. A UNIFORM SEMICIRCULAR PLATE.—In the figure (Fig. 60) it is clear that G must lie upon the symmetrical radius OC.

Let OG $= \bar{y}$; and suppose we cut the semicircle into rods such as PQRS, parallel to the base AB. Let y = OT and let the radius be a. Then the thickness of a rod is δy, and its length

$$TQ = \sqrt{(OQ^2 - OT^2)} = \sqrt{a^2 - y^2}.$$

Its mass $= 2\rho\sqrt{a^2 - y^2}\,.\,\delta y$, where $\rho =$ the mass of unit area. The first moment about AB

$$= 2\rho\sqrt{a^2 - y^2}\,\delta y \times y.$$

Hence

$$\begin{aligned} M\bar{y} &= 2\int_{y=o}^{y=a} \rho\sqrt{a^2 - y^2}\,ydy \\ &= \rho\int_{y=o}^{y=a} \sqrt{a^2 - y^2}\,.\,2\,ydy \\ &= \rho\int_{y=o}^{y=o} \sqrt{a^2 - y^2}\,.\,\frac{dy^2}{dy}\,.\,dy\,^* \\ &= \rho\int_{y=o}^{y=a} \sqrt{a^2 - y^2}\,dy^2 \\ &= -\rho\int_{y=o}^{y=a} (a^2 - y^2)^{\frac{1}{2}}\,d\,(a^2 - y^2) \\ &= -\rho\left[\frac{(a^2 - y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_{y=o}^{y=a} \\ &= -\rho\,(0 - \tfrac{2}{3}\,a^3) \\ &= \tfrac{2}{3}\,\rho a^3. \end{aligned}$$

But $M = \rho\,.\,\frac{1}{2}\,\pi a^2$.

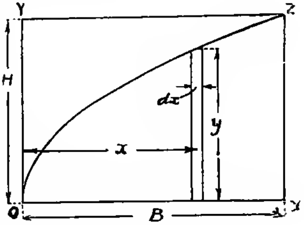

FIG. 61.

Hence

$$OG = \bar{y} = \frac{\frac{2}{3}\,\rho a^3}{\frac{1}{2}\,\pi\rho a^2} = \frac{4a}{3\pi}.$$

2°. A PARABOLA.—Consider next the parabola

---

[*] This is an example of the method explained in § 43.

$y^2 = 4\,ax$, and take the area between the curve and the axis of x.

$$\text{Area of curve} = \int y\,dx = \int 2\,a^{\frac{1}{2}}x^{\frac{1}{2}}dx$$

$$= 2a^{\frac{1}{2}}\int_{\text{O}}^{\text{B}} x^{\frac{1}{2}}dx = \left[2\,a^{\frac{1}{2}}\,.\,\frac{2}{3}\,x^{\frac{3}{2}}\right]_{\text{O}}^{\text{B}}$$

$$= \frac{4}{3}\,a^{\frac{1}{2}}\text{B}^{\frac{3}{2}}.$$

Now $2\,a^{\frac{1}{2}}\text{B}^{\frac{1}{2}} = \text{H}$.

$$\therefore \text{Area of curve} = \frac{2}{3}\,\text{BH}.$$

$$\text{First moment about OY} = \int xy\,dx = \int 2\,a^{\frac{1}{2}}x^{\frac{3}{2}}dx$$

$$= 2\,a^{\frac{1}{2}}\int_{\text{O}}^{\text{B}} x^{\frac{3}{2}}dx = \left[2\,a^{\frac{1}{2}}\,.\,\frac{2}{5}\,x^{\frac{5}{2}}\right]_{\text{O}}^{\text{B}}$$

$$= \frac{4}{5}\,a^{\frac{1}{2}}\text{B}^{\frac{5}{2}} = \frac{2}{5}\,\text{B}^2\text{H}.$$

$$\therefore \text{distance of centroid from OY} = \frac{\frac{2}{5}\,\text{B}^2\text{H}}{\frac{2}{3}\text{BH}} = \frac{3}{5}\,\text{B}.$$

To find the distance of the centroid from OX, note that the centroid of each strip is at its middle point.

Hence first moment of strip about OX

$$= \int_{x=o}^{x=\text{B}} \frac{y}{2}\,.\,y\,dx = \tfrac{1}{2}\int_{o}^{\text{B}} 4\,ax\,dx$$

$$= 2a\int_{o}^{\text{B}} x\,dx = 2a\left[\frac{x^2}{2}\right]_{o}^{\text{B}} = a\text{B}^2.$$

$\therefore$ distance of centroid from OX

$$= \frac{a\text{B}^2}{\frac{2}{3}\,\text{BH}} = \frac{3a\text{B}}{2\text{H}} = \frac{3}{2\text{H}}\,.\,\frac{\text{H}^2}{4} = \frac{3\text{H}}{8}.$$

Surfaces of Revolution.—The following rules with regard to surfaces of revolution (i.e. surfaces generated by revolving a plane curve about an axis), commonly known as the "Theorems of Pappus," follow directly from the consideration of centroids.

1. *The volume of a surface of revolution is equal to the area of the surface multiplied by the length of the path of its centroid.*

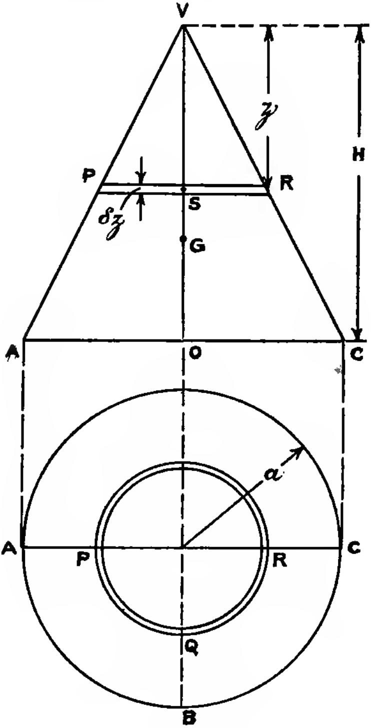

FIG. 62.

2. *The surface of a surface of revolution is equal to the length of the generating arc multiplied by the length of the path of its centroid.*

3°. A UNIFORM RIGHT CIRCULAR CONE.—In our general discussion, we considered only flat plates. The extension to solid bodies is easy, but here we must consider first moments not about axes but *about planes.*

Let OV be the axis of the cone, V being the vertex: and let any distance measured along the axis VS $= z$.

Cut the cone into slices such as PSR of thickness δz, parallel to the base ABC. Then the centre of gravity (S) of each plate lies on the axis VO, and hence the centre of gravity G of the cone lies on VO.

Suppose then the mass of each plate to be concentrated at its centre of gravity, and take moments about a plane through V parallel to the base.

If a = radius of base,
h = height OV,
ρ = mass per unit volume (density),
then

$$\text{the radius of a plate} = \text{SR} = \text{OC}\,\frac{\text{VS}}{\text{VO}} = a\,\frac{z}{h};$$

$$\text{the area of the plate} = \pi\left(a\,\frac{z}{h}\right)^2 = \pi\,\frac{a^2}{h^2}\,z^2;$$

$$\therefore \text{ its mass} = \pi\,\frac{a^2}{h^2}\,z^2\delta z\rho.$$

$\therefore$ The first moment

$$= \int_o^h \pi\,\frac{a^2}{h^2}\,z^2\,.\,z\,.\,\rho\delta z$$

$$= \pi\rho\,\frac{a^2}{h^2}\int_o^h z^3dz = \pi\rho\,\frac{a^2}{h^2}\,.\,\frac{h^4}{4} = \frac{\pi\rho}{4}\,a^2h^2.$$

And the mass of the cone

$$= \tfrac{1}{3}\text{ area of base} \times \text{height} \times \text{density}$$

$$= \tfrac{1}{3}\,\pi a^2\,.\,h\,.\,\rho = \frac{\pi\rho}{3}\,a^2h.$$

Hence

$$\frac{\pi\rho}{3}a^2h \,.\, \mathrm{VG} = \frac{\pi\rho}{4}a^2h^2$$

Or $$\mathrm{VG} = \tfrac{3}{4}\,h = \tfrac{3}{4}\,\mathrm{VO}.$$

§ 51. Moment of Inertia and Radius of Gyration.—DEFINITION.—The *moment of inertia* or *second moment* of a particle m situated at P *about an axis* or line AB is

$$m\mathrm{PN}^2 = my^2,$$

where y is the perpendicular distance of P from AB.

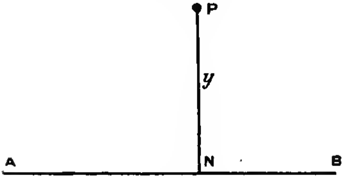

FIG. 63.

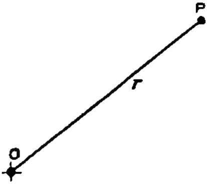

FIG. 64.

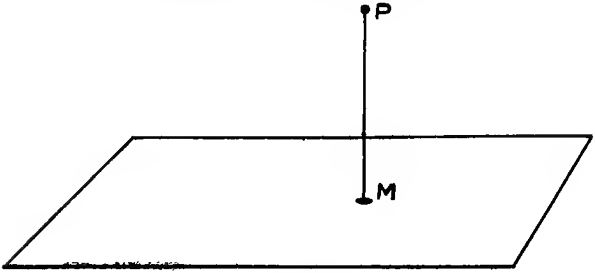

FIG. 65.

The *second moment* of a particle m at P about a *point* O or axis through O perpendicular to the plane of the paper is

$$m \,.\, \mathrm{OP}^2 = mr^2$$

This is often called a *polar second moment.*

The second moment of a particle m at P about a *plane* is similarly

$$m\mathrm{PM}^2 = mz^2,$$

where z is the perpendicular distance of P from the plane.

The second moment of a system of particles is the sum of the second moments of the particles taken separately. This applies equally to a continuous solid body which can be supposed sub-divided into a large number of small parts each of which is to be treated as a particle.

The *polar moment of inertia of a wheel*, of radius a and mass M (the mass of the spokes and axle being neglected) is clearly $\mathrm{M}a^2$, every point of the rim being at the same distance from the axis.

MOMENT OF INERTIA OF AN AREA.—Although an area has no mass and can therefore strictly have no inertia, the second moment of an area, i.e. the sum of the products of each element of area by the square of its distance from a line or axis, is of great importance in a number of problems upon beams and shafts and is usually called the *moment of inertia of the area*. All the properties that we prove for masses apply equally well to areas.

RADIUS OF GYRATION.—In the same way that in considering first moments of bodies or areas we can regard the whole area as concentrated at a point called the centre of gravity or centroid, we can also in dealing with moments of inertia regard the whole area as concentrated at a point which we call the *secondroid* which will depend upon the axis taken. Then the distance of this point from the axis or line about which moments have been taken is called the *radius of gyration about the axis* and is given the letter k. The value of k can be found only by the relation

$$Mk^2 \text{ or } Ak^2 = I$$

$$\text{i.e. } k = \sqrt{\frac{I}{M}} \text{ or } \sqrt{\frac{I}{A}}.$$

The first form is used for solid bodies and the second for areas.

The radius of gyration is a very important quantity and comes into a very large number of problems in constructional design as well as in problems dealing with rotating and vibrating bodies.

§ 52. **Two Properties of Second Moments.**—1°. (a) Consider the second moments of a system :—

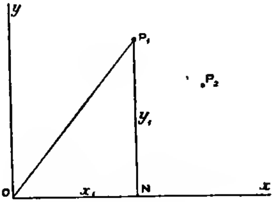

FIG. 66.

(1) about an axis OX,

$$\begin{aligned} I_X &= m_1y_1^2 + m_2y_2^2 + \ . \ . \ . \\ &= \Sigma my^2 \end{aligned}$$

(where Σ means " the sum of terms like ").

(2) about a perpendicular axis OY

$$\begin{aligned} I_Y &= m_1x_1^2 + m_2x_2^2 + \ . \ . \ . \\ &= \Sigma mx^2. \end{aligned}$$

(3) about their point of intersection O,

$$\begin{aligned} J &= m_1r_1^2 + m_2r_2^2 + \ . \ . \ . \\ &= \Sigma mr^2. \end{aligned}$$

Then

$$r_1^2 = x_1^2 + y_1^2,\ r_2^2 = x_2^2 + y_2^2,\ \ldots$$

and

$$\begin{aligned} J &= \Sigma mr^2 = \Sigma m\,(x^2 + y^2) \\ &= \Sigma(mx^2 + my^2) \\ &= \Sigma mx^2 + \Sigma my^2 = I_X + I_Y. \end{aligned}$$

If k_X, k_Y and k_P are the radii of gyration about OX, OY and O respectively we have $J = Mk_P^2$; $I_X = Mk_X^2$ and $I_Y = Mk_Y^2$ $\therefore k_P^2 = k_X^2 + k_Y^2$.

Similarly it is proved that

The second moment about an axis (called also polar second moment in the case of a plane area) is equal to the sum of the second moments about any two perpendicular planes intersecting in the axis.

(*b*) Next consider three perpendicular axes :—

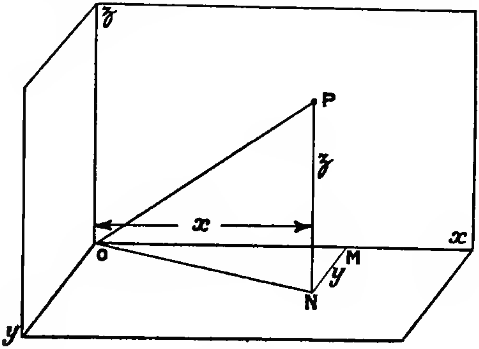

FIG. 67.

OX, horizontal in the plane of the paper.
OY, horizontal perpendicular to the plane of the paper.
OZ, vertical in the plane of the paper.

Let

I_P = second moment about O
I_X, I_Y, I_Z = second moments about OX, OY, OZ (respectively)

A, B, C = second moments about the planes YOZ, ZOX, XOY (respectively).

Then clearly

$$I_X = B + C,\ I_Y = C + A,\ I_Z = A + B.$$

It can also be proved that

$$I_P = A + B + C = \tfrac{1}{2}(I_X + I_Y + I_Z)$$

$$\therefore k_P^2 = \tfrac{1}{2}(k_X^2 + k_Y^2 + k_Z^2).$$

2°. Parallel Axes.—If

I = the moment of inertia of a plate about any axis AB in its plane,

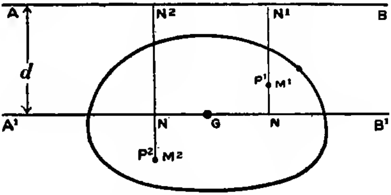

Fig. 68.

I_c = the moment of inertia about a parallel axis A′B′ through its centre of gravity G,

d = perpendicular distance of G from AB,

M = mass of plate,

then

$$I = I_c + Md^2.$$

For (see Fig. 68) and let $P^1 N^1 = y_1$ and $P^2 N^2 = y_2$; $P^1N = y_1'$ and $P^2N = y_2'$

$$I = m_1y_1^2 + m_2y_2^2 + \ \ldots\ = Smy^2$$
$$= m_1(y_1' + d)^2 + m_2(y_2' + d)^2 + \ \ldots\ = Sm(y_1' + d)^2$$
$$= Sm(y_1'^2 + 2dy_1' + d^2)$$
$$= Smy_1'^2 + 2Smdy_1' + Smd^2$$
$$= Smy_1'^2 + 2dSmy_1' + d^2Sm$$
$$= I_c + 2d \cdot 0^* + d^2 \cdot M$$
$$= I + Md^2$$

---

[*] This is 0 because the sum is the first moment of the system about an axis through the centre of gravity which is 0 (§ 48, p. 146).

Note.—The perpendicular is considered positive when drawn *upwards*, so that (in the figure) P_2N is positive, while P_1N is negative, and always

$$y_1 = y_1' + d.$$

Exactly the same formula expresses the moment of inertia of a solid body about any axis in terms of a parallel axis through the centre of gravity.

Similarly for areas we have

$$I = I_c + Ad^2.$$

Corresponding to this we have for solids

$$Mk^2 = Mk_c^2 + Md^2$$

and for areas

$$Ak^2 = Ak_c^2 + Ad^2$$

Both these give

$$k^2 = k_c^2 + d^2.$$

It will thus be seen clearly that the radius of gyration is a *length* and the formula comes the same for solid bodies as for areas.

The dimensions of moments of inertia.—Many students find considerable difficulty in getting clear ideas upon moments of inertia; this is, we think, largely because they cannot form a concrete idea of what dimensions the quantity is. In the case of areas the moment of inertia is of the order area × $(\text{length})^2 = (\text{length})^4$ and it is not easy to get a concrete idea of this dimension; in the case of solids it is of the order mass × $(\text{length})^2$ and this again is not a familiar dimension although it is much easier to conceive. When difficulty of this kind is experienced it is best to regard the moment of inertia as some new idea and not to attempt to form a concrete idea of it as being of the dimension $(\text{length}).^4$

§ 53. Some Important Moments of Inertia.

1°. UNIFORM ROD.—*Axis through the centre perpendicular to the rod.*—Let the length be l, and the mass M. Subdivide the rod into small lengths $PQ = \delta x$, x being OP.

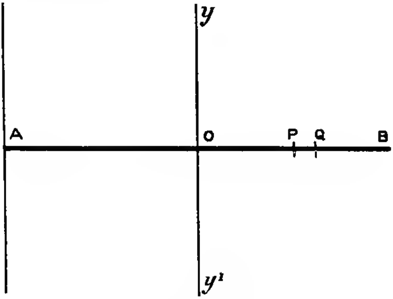

FIG. 69.

The mass per unit length

$$\rho = \frac{M}{l}.$$

Hence the mass of $PQ = \frac{M}{l}\,dx.$

$$I = \text{Moment of inertia} = 2\int_0^{\frac{l}{2}} x^2 \,.\, \frac{M}{l}\,dx.$$

$$= \frac{2M}{l}\int_0^{\frac{l}{2}} x^2 dx$$

$$= \frac{2M}{l}\,\tfrac{1}{3}\left(\frac{l}{2}\right)^3 = \frac{Ml^2}{12}.$$

[The integral gives the moment of inertia of one half, OB, of the rod: this is why it is doubled.]

Moment of inertia about axis through A perpendicular to the rod.

We make use of 2° in the last §.

$$I = I_c + Md^2 = \frac{Ml^2}{12} + \frac{Ml^2}{4} = \frac{Ml^2}{3}.$$

2°. UNIFORM RECTANGULAR PLATE.—Let the mass $= M$ and the sides be of length B and H.

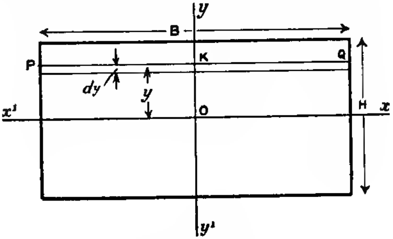

FIG. 70.

Divide the rectangle into rods such as PQ of thickness dy, parallel to $X'X$, y being OK.

Then the mass of a rod $= \frac{M}{H} \cdot dy$.

And M. I. (moment of inertia) about $X'X = I_X$

$$= 2\int_0^{\frac{H}{2}} \cdot \frac{M}{H} \cdot \; y^2 dy = \frac{2M}{H} \cdot \left(\frac{H}{2}\right)^3 \cdot \frac{1}{3}$$

$$= \frac{MH^2}{12}.$$

In fact this is really the same case as that of the rod.

The M. I. about $Y'Y = I_Y = \frac{MB^2}{12}$. And by § 52, 1° the polar M. I. *about axis through O perpendicular to the rectangle*

= sum of M. I.'s about XX' and $Y'Y$

$$= \frac{M}{12}(B^2 + H^2).$$

Similarly for the rectangular *area* we have, since the area $A = BH$ corresponds to M,

$$I_X = \frac{BH^3}{12}$$

$$I_Y = \frac{HB^3}{12}$$

$$J = \frac{A}{12}(B^2 + H^2).$$

3°. THIN WHEEL (spokes and axle neglected).

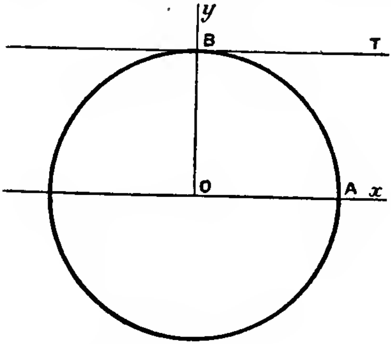

Fig. 71.

Let M = mass.
R = radius.

Then J (polar moment of inertia) about the axis is

$$J = MR^2.$$

This is the same clearly as the M. I. about its centre O.

To find the M. I. about a *diameter*.

We have (§ 52, 1°)

$$J = I_X + I_Y$$

(OX and OY being perpendicular)

$$= 2\,(\text{M. I. about OX})$$

$\therefore$ M. I. about OX $= \frac{1}{2}MR^2$.

Using § 52, 2°,

M. I. about a *tangent* BT

$$= \tfrac{1}{2}MR^2 + MR^2 = \tfrac{3}{2}MR^2,$$

4°. UNIFORM CIRCULAR PLATE.—Let the radius $= \mathrm{R}$.

To find the M. I. about an axis through the centre O perpendicular to the plate, divide the plate up into thin circular strips as shown in the figure.

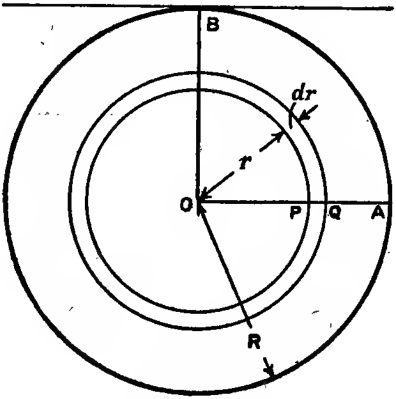

FIG. 72.

If r = radius OP of any strip, and dr = its thickness PQ,

Then its mass = density × length × thickness

$$= \frac{\mathrm{M}}{\pi \mathrm{R}^2} \times 2\pi r \times r$$

$$= \frac{2\mathrm{M}}{\mathrm{R}^2} r\delta r.$$

$$dJ = \frac{2\mathrm{M}}{\mathrm{R}^2} r\delta r \times r^2 = \frac{2\mathrm{M}}{\mathrm{R}^2} r^3 \delta r.$$

$$\therefore \text{M. I. of plate} = \int_o^{\mathrm{R}} \frac{2\mathrm{M}}{\mathrm{R}^2} r^3 dr$$

$$= \mathrm{J} = \frac{2\mathrm{M}}{\mathrm{R}^2} \int_o^{\mathrm{R}} r^3 dr$$

$$= \frac{2\mathrm{M}}{\mathrm{R}^2} \cdot \frac{\mathrm{R}^4}{4} = \tfrac{1}{2}\,\mathrm{MR}^2.$$

To find the M. I. about a diameter Fig. 72, we have as in 3° (by § 52, 1°)

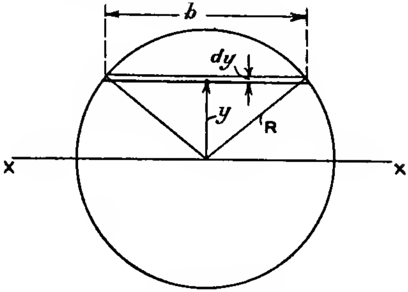

FIG. 73.

$$\tfrac{1}{2}\ \mathrm{MR}^2 = 2\ (\text{M. I. about diameter})$$

$$\therefore\ \text{M. I. } \textit{about diameter} = \tfrac{1}{4}\ \mathrm{MR}^2.$$

And again

$$\text{M. I. about tangent (by § 52, 2°)}$$
$$= \tfrac{1}{4}\ \mathrm{MR}^2 + \mathrm{MR}^2 = \tfrac{5}{4}\ \mathrm{MR}^2.$$

For a circular area we shall have about the diameter

$$\mathrm{I} = \frac{\pi \mathrm{R}^4}{4}$$
$$= \frac{\pi \mathrm{D}^4}{64}$$

where D is the diameter.

The value of the moment of inertia of a circle about its diameter may also be obtained at much greater length as follows; the student should complete it as an exercise.

Referring to Fig. 73

$$\mathrm{I}_{\mathrm{XX}} = 2\int_0^{+\mathrm{R}} b\,.\,dy\,.\,y^2.$$

Now let θ be the angle between the normal y and the radius R.

Then
$$b = 2\text{ R}\sin\theta$$
$$y = \text{R}\cos\theta$$
$$\therefore \frac{dy}{d\theta} = -\text{ R}\sin\theta$$
$$\therefore \int_0^{+\text{R}} b \,.\, dy \,.\, y^2 = \int_{\frac{\pi}{2}}^{0} \text{R}^2\cos^2\theta \,.\, 2\text{ R}\sin\theta\,(-\text{ R}\sin\theta)d\theta$$
$$= 2\text{ R}^4\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta d\theta$$

Calling the integral B

$$\text{we have B} = \tfrac{1}{4}\int_0^{\frac{\pi}{2}}(2\sin\theta\cos\theta)^2 d\theta$$
$$= \tfrac{1}{4}\int_0^{\frac{\pi}{2}}\sin^2 2\theta\, d\theta$$
$$= \tfrac{1}{8}\int_0^{\frac{\pi}{2}}(1 - \cos 4\theta)d\theta$$
$$= \tfrac{1}{8}\Big[\theta - \tfrac{1}{4}\sin 4\theta\Big]_0^{\frac{\pi}{2}}$$
$$= \frac{\pi}{16}$$
$$\text{whence I} = 2 \,.\, 2\text{R}^4 \,.\, \frac{\pi}{16}$$
$$= \frac{\pi\text{R}^4}{4}$$

as before.

The reader will find that he can avoid a great deal of integration by the judicious use of the propositions given in § 52 which are often applicable to symmetrical bodies.

5°. Sphere.—A similar method applies to a uniform sphere which is subdivided into spherical shells, We find

$$\text{M. I. about diameter} = \tfrac{2}{5}\,MR^2$$
$$\text{M. I. about tangent line} = \tfrac{7}{5}\,MR^2$$

6°. Uniform Cylinder.—Let the length be 2L, and the radius be R. To find M. I. about axes.

Take three perpendicular axes:—

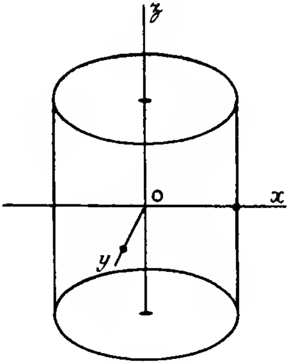

Fig. 74.

Oz, the axis of the cylinder, Ox, Oy, two axes at right angles through the centre perpendicular to the axis, Fig. 74.

Then using the notation of (§ 52, 1°)

$$A = B = \tfrac{1}{4}\,MR^2 \text{ by } 4^\circ$$

for this comes to taking the M. I. of a circular plate about a diameter]

$$\text{and } C = \tfrac{1}{3}\,ML^2 \text{ by } 1^\circ \text{ similarly,}$$

we have at once

$$\text{M. I. about axis OZ} = I_z = \tfrac{1}{2}\,MR^2$$
$$\text{and M. I. about OX} = I_x = B + C$$
$$= \tfrac{1}{4}\,MR^2 + \tfrac{1}{3}\,ML^2$$
$$= M\left(\tfrac{1}{4}\,R^2 + \tfrac{1}{3}\,L^2\right).$$

7°. Parabolic Segment.—Referring to Fig. 61,

$$\text{Second moment about OY} = \int x^2 y\,dx = \int 2a^{\frac{1}{2}}x^{\frac{5}{2}}dx$$

$$= 2a^{\frac{1}{2}}\int_0^B x^{\frac{5}{2}}dx = \left[2\,a^{\frac{1}{2}}\;.\;\frac{2}{7}x^{\frac{7}{2}}\right]_0^B$$

$$= \frac{4}{7}\;a^{\frac{1}{2}}B^{\frac{7}{2}} = \frac{2}{7}\;B^3H$$

$$\therefore\ k_Y{}^2 = \frac{\frac{2}{7}\,B^3H}{\frac{2}{3}\,BH},$$

$$\text{or } k_Y\ =\ \sqrt{\tfrac{3}{7}}\,B.$$

If the second moment is required about the base XZ, we proceed as follows :—

$$I_{OY} = \frac{2}{7}\,B^3H.$$

$$I_{CC} = I_{OY} - A\ .\ d^2$$

$$= \frac{2}{7}HB^3 - \frac{2}{3}\,.\ BH\ .\ \frac{9B^2}{25}$$

$$I_{XZ} = I_{CC} + Ad_1{}^2$$

$$= \frac{2}{7}HB^3 - \frac{6}{25}\,HB^3 + \frac{8}{75}\,HB^3$$

$$= HB^3\left\{\frac{150 - 126 + 56}{525}\right\} = HB^3\,.\,\frac{80}{525}$$

$$= \frac{16}{105}\;HB^3.$$

§ 54. **Product Moments.**—If dm is an element of mass at distances x, y from two perpendicular axes the quantity $\int xydm$ is called the *product moment,* which in the case of areas becomes $\int xyda$. This quantity bears some resemblance to the moment of inertia and is useful in some more advanced problems in connexion with beams whose sections have no axis of symmetry.

It follows at once that the product moment about an axis of symmetry is zero because corresponding to every point on one side of such axis there is a another point on the other side whose ordinate is equal to it numerically but opposite in sign.

§ 55. **Centre of Pressure and Load Point (C).**—In the case of bodies subjected to fluid pressure such as dams, sluices, dock-gates, etc., and those subjected to a uniformly varying stress such as beam sections, we require to find the *centre of pressure*, i.e. the point L at which a single force equal to the total pressure or stress upon the body may act to keep it in equilibrium.

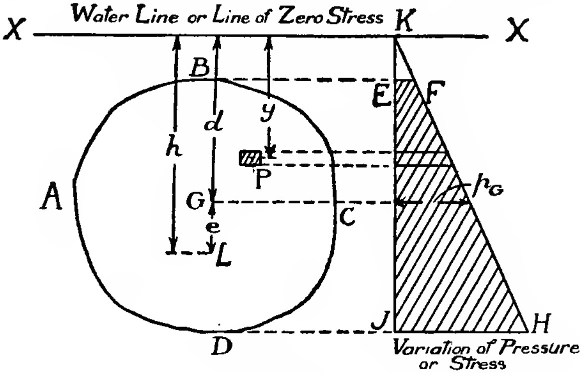
X
Water Line or Line of Zero Stress
K
X
B
E
F
d
y
h
P
A
G
p_G
C
e
L
J
H
D
Variation of Pressure or Stress

FIG. 75.

In the case of stresses L is often called the *load point.* The depth of the centre of pressure is usually measured from the line XX of zero pressure or stress which is the *water line* in the case of fluid pressure and the *neutral axis* in the case of beams.

Referring to Fig. 75, if *da* is an element of area situated around the point P, since the intensity of pressure or stress is proportional to the depth below XX, the intensity of pressure at P $= p_\rho = \rho \cdot y$ where ρ is some constant, which in the case of fluid pressure is equal to the weight of unit volume of the fluid and in the case of stresses is the intensity of stress at unit distance from the neutral axis.

$$\text{The force on element of area} = dQ = p_\rho \cdot da$$
$$= \rho y da.$$
$$\therefore \text{ total force on body} = Q = \int \rho y da$$
$$= \rho \int y da$$

$= \rho \times$ first moment about XX

$= \rho \cdot A \cdot d = A\rho d \quad . \quad . \quad . \quad (1)$

= area × intensity of pressure or stress at centroid.

Now take moments about the line XX.

$$\text{Moment of force on element} = p_\rho \cdot y da$$
$$= \rho y^2 da.$$

$\therefore$ total moment about XX $= \int \rho y^2 da = \rho I_x$.

$\therefore$ if h is the depth of the centre of pressure

$$Q \cdot h = \rho I_x$$

$$\text{i.e. } h = \frac{\rho I_x}{Q} = \frac{\rho I_x}{\rho \cdot Ad}$$

$$= \frac{I_x}{Ad} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$= \frac{\text{second moment of area about XX}}{\text{first moment of area about XX}}.$$

In some cases it is more convenient to find the distance e between the centroid and the centre of pressure. We then use the relation (see p. 157)

$$I_x = I_c + Ad^2.$$

Equation (2) then becomes

$$h = d + e = \frac{I_c + Ad^2}{Ad}$$

$$= \frac{I_c}{Ad} + d$$

$$\therefore e = \frac{I_c}{Ad} = \frac{Ak_c^2}{Ad}$$

$$= \frac{k_c^2}{d} \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

In some problems on stress distribution we are given the position of the load point and require to find the distance d from the centroid to the neutral axis; we then use the relation

$$d = \frac{k_c^2}{e} \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

In such problems e is called the *eccentricity* of the load.

Considering the application to stresses we notice that as e, the eccentricity, diminishes, d gets longer, i.e. the neutral axis gets farther away and as a result of this the difference between the maximum and minimum stresses JH and EF will diminish, the stresses ultimately becoming uniform across the section as L coincides with G. The neutral axis can then be regarded as at an infinite distance away; this will be seen mathematically by making e very small in equation (4), because k_c^2 is a finite quantity and a finite quantity when divided by a very small quantity gives a very great quantity.

§ 56. **Special Cases of Centre of Pressure.**—1. *Rectangular Surface Extending up to the Water Line.*—Referring to p. 160 if H is the depth below the water line and B the breadth

$$I_x = \frac{BH^3}{3},\ A = BH \text{ and } d = \frac{H}{2}.$$

$$\therefore h = \frac{I_x}{Ad} = \tfrac{1}{3}BH^3/\tfrac{1}{2}BH^2 = \frac{2H}{3}.$$

Therefore in dock-gates, dams, etc., the centre of pressure acts at $\frac{2}{3}$ depth from the water line.

Similarly in reinforced concrete rectangular beams the resulting compression stress is at $\frac{2}{3}$ distance from the neutral axis to the compression edge.

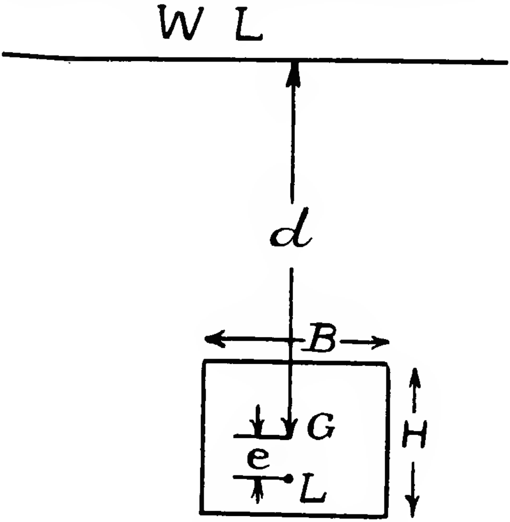

FIG. 76.

2. *Rectangular Surface Entirely below the Water Line.*—k_c^2 for a rectangle $= \frac{H^2}{12}$ [see p. 160].

$$\therefore \text{ in this case } e = \frac{H^2}{12d}.$$

It will be noted that e diminishes as d increases or that the centre of pressure or load point gets nearer to the centroid of the section as the depth of the upper portion of the area below the water line increases. In the limiting case in which $d = \frac{H}{2}$, $e = \frac{H}{6}$, i.e. $h = \frac{H}{2} + \frac{H}{6} = \frac{2H}{3}$, thus agreeing with the previous case as, of course, it should.

3. *Circular Surface Entirely Below the Water Line.*—I for a circle about its diameter $= \frac{\pi D^4}{64}$ [p. 163].

$$\therefore k_\alpha^2 = \frac{D^2}{16}$$

$$\therefore \text{in this case } e = \frac{D^2}{16d}.$$

This will be less than for a square surface of side equal to D.

§ 57. (M) **Friction on Footstep Bearings.**—*Flat Bearing.*—We will now consider the application of integration to calculate the power absorbed in footstep bearings, Fig. 77. In the case of the flat bearing we will assume that the pressure is uniformly distributed over the whole bearing

$$\therefore \text{intensity of pressure} = p = \frac{\text{load}}{\text{area}} = \frac{W}{\pi R^2}$$

∴ pressure on an elementary ring of width dr situated at a distance r from the centre

$$= dW = p \times \text{area of ring} = p \times 2\pi r dr$$

$$\therefore \text{friction on ring} = \mu dW = \mu p \,.\, 2\pi r dr$$

∴ moment of friction on ring about centre

$$= dM_f = \mu dW \,.\, r = 2\pi\mu p r^2 dr$$

∴ moment of friction on whole surface about centre

$$= M_f = \int_0^R 2\pi\mu p r^2 dr$$

$$= 2\pi\mu p \int_0^R r^2 dr$$

$$= 2\pi\mu p \left[\frac{r^3}{3}\right]_0^R$$

$$= \frac{2\pi\mu p R^3}{3} = \frac{2\mu W R}{3}$$

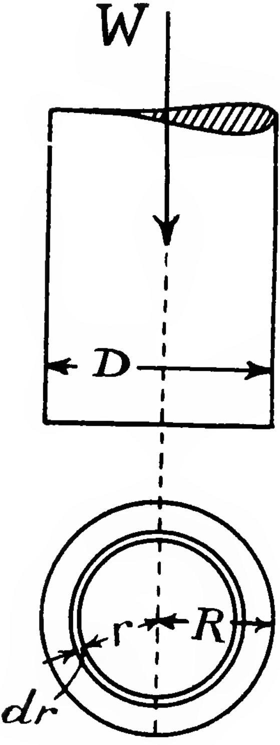

FIG. 77.

∴ work lost in friction per second if n is the number of revolutions per second $= 2\pi n \mathrm{M}_f$ (by the rule that work per second $= 2\pi n \times$ moment of force) and R is in inches

$$= \frac{4\pi\mu \mathrm{WR}n}{3} \text{ inch-lb.} \quad . \quad . \quad . \qquad (1)$$

$$\therefore \text{ Horse-power absorbed } = \frac{4\pi\mu\mathrm{WR}n}{3 \times 550 \times 12}$$

$$= \frac{\mu\mathrm{WR}n}{1575} \quad . \quad . \quad . \qquad (2)$$

Collar Bearing.—In this case referring to Fig. 78 we get by exactly similar reasoning

$$\mathrm{M}_f = 2\pi\mu p \left[\frac{r^3}{3}\right]_{\mathrm{R}_2}^{\mathrm{R}_1}$$

$$= \frac{2\pi\mu p\,(\mathrm{R}_1{}^3 - \mathrm{R}_2{}^3)}{3}.$$

The area in this case $= \pi\,(\mathrm{R}_1{}^2 - \mathrm{R}_2{}^2)$

$$\therefore p = \frac{\mathrm{W}}{\pi\,(\mathrm{R}_1{}^2 - \mathrm{R}_2{}^2)}$$

$$\therefore \mathrm{M}_f = \frac{2\pi\mu\mathrm{W}}{3}\,\frac{(\mathrm{R}_1{}^3 - \mathrm{R}_2{}^3)}{\pi\,(\mathrm{R}_1{}^2 - \mathrm{R}_2{}^2)}$$

$$= \frac{2\mu\mathrm{W}}{3}\,\frac{(\mathrm{R}_1 - \mathrm{R}_2)\,(\mathrm{R}_1{}^2 + \mathrm{R}_1\mathrm{R}_2 + \mathrm{R}_2{}^2)}{(\mathrm{R}_1 - \mathrm{R}_2)\,(\mathrm{R}_1 + \mathrm{R}_2)}$$

$$= \frac{2\mu\mathrm{W}}{3}\,\frac{(\mathrm{R}_1{}^2 + \mathrm{R}_1\mathrm{R}_2 + \mathrm{R}_2{}^2)}{(\mathrm{R}_1 + \mathrm{R}_2)} \quad . \quad . \quad . \qquad (3)$$

∴ Horse-power absorbed

$$= \frac{\mu\mathrm{W}\,(\mathrm{R}_1{}^2 + \mathrm{R}_1\mathrm{R}_2 + \mathrm{R}_2{}^2)n}{1575\,(\mathrm{R}_1 + \mathrm{R}_2)} \quad . \quad . \quad . \qquad (4)$$

In the case where the collar is very small and R_1 and R_2 become practically equal, each say equal to R,

$$\mathrm{M}_f = \frac{2\mu\mathrm{W}}{3} \cdot \frac{3\mathrm{R}^2}{2\mathrm{R}} = \mu\mathrm{WR}.$$

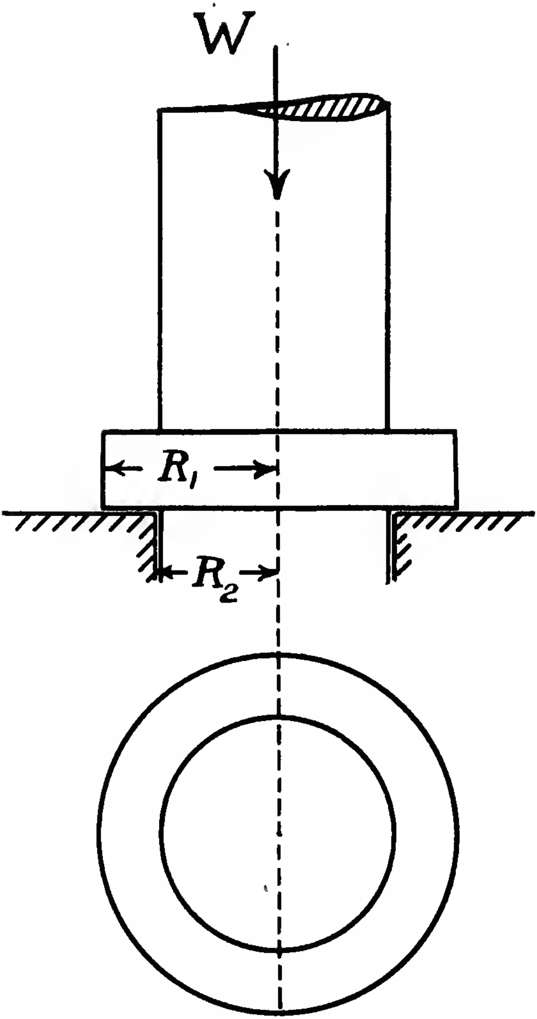

FIG. 78.

Conical Bearing.—Referring to Fig. 79, the normal pressure p if assumed constant will be such that

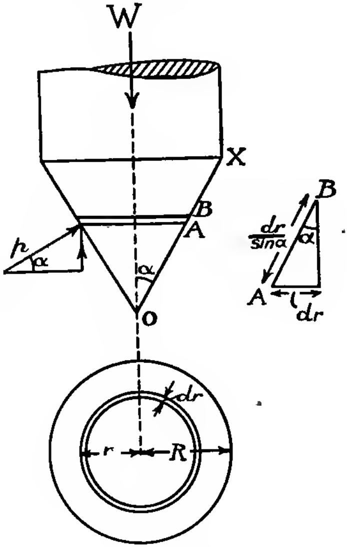

FIG. 79.

p × area of cone $= \dfrac{W}{\sin a}$ and area of cone = OX × ½ circumference of base

$$\therefore p \times \pi R \times OX = \frac{W}{\sin a}$$

$$\text{i.e. } p \times \pi R \times \frac{R}{\sin a} = \frac{W}{\sin a}$$

$$\therefore p = \frac{W}{\pi R^2}.$$

This is independent of the angle of the cone and is the same as for the flat pivot.

$\therefore$ pressure on ring AB $= p \times 2\pi r$ AB.

$$= \frac{p \times 2\pi r dr}{\sin \alpha}.$$

$\therefore$ friction on AB acting at right angles to the paper

$$= \mu p \times \frac{2\pi r dr}{\sin \alpha}.$$

$\therefore$ moment about centre line

$$= dM_f = \frac{\mu p \times 2\pi r dr \, . \, r}{\sin \alpha}$$

$\therefore$ total moment of friction about the centre line

$$= M_f = \int_0^R \mu p \, . \, \frac{2\pi r^2 dr}{\sin \alpha}$$

$$= \frac{\mu p \, . \, 2\pi}{\sin \alpha} \int_0^R r^2 dr$$

$$= \frac{\mu \, . \, p \, . \, 2\pi \, . \, R^3}{\sin \alpha \, . \, 3}$$

$$= \frac{\mu \, . \, W \, . \, 2\pi \, R^3}{\pi \, . \, R^2 \, 3 \sin \alpha}$$

$$= \frac{2\mu \, . \, W \, . \, R}{3 \sin \alpha} \text{ inch-lb.} \quad . \quad . \quad (5)$$

$\therefore$ Horse-power absorbed $= \dfrac{\mu \, . \, W \, Rn}{9900 \sin \alpha} \quad . \quad (6)$

For a truncated cone of maximum and minimum radii R_1, R_2 respectively, we get

$$M_f = \frac{\mu p \, . \, 2\pi}{\sin \alpha} \int_{R_2}^{R_1} r^2 dr$$

$$= \frac{\mu \, . \, p \, . \, 2\pi}{\sin \alpha} \left(\frac{R_1^3 - R_2^3}{3} \right)$$

$$= \frac{\mu \, . \, W \, . \, 2\pi \, (R_1^3 - R_2^3)}{3\pi (R_1^2 - R_2^2) \sin \alpha}$$

$$= \frac{\mu 2W (R_1^2 + R_1 R_2 + R_2^2)}{3 \sin \alpha \, (R_1 + R_2)} \quad . \quad . \quad . \quad (7)$$

∴ Horse-power absorbed

$$= \frac{\mu W(R_1^2 + R_1R_2 + R_2^2)n}{9900 \sin \alpha \, (R_1 + R_2)} \quad . \quad . \quad (8)$$

When $\alpha = 90°$ this gives, as we should expect, the same result as for the collar bearing.

Schiele's Pivot.—In the pivots considered up to the present the wear will be greatest at the outside because the velocity is greatest there, thus causing considerable unevenness in wear.

Schiele's pivot is made of such a shape that the wear is theoretically equal if equal pressure be assumed as before.

We will now determine the shape of the curve which will give this result.

Consider an elementary strip (Fig. 80) of the bearing. The load on the strip will be the proportion of W carried by a circular ring of thickness AC = dr.

$$\text{Load} = \frac{W}{A} \, . \, 2\pi r \, . \, dr,$$

where A is the horizontal projection of the area over which the weight is carried.

Assuming that the pressure is constant and equal to p we have

$$\therefore \; p \times AB \, . \; 2\,\pi r \sin \theta = \frac{W}{A} \, . \, 2\pi r dr$$

As in the case of the conical pivot we shall have moment of friction on element

$$dM_f = \frac{\mu p \, . \, 2\pi \, . \, r^2 dr}{\sin \theta}.$$

The normal wear ac of the pivot at any strip, in the direction of p, will be proportional to the work done against the friction, if AB is the same for each strip because the friction is the force resisting wear; so that work done in wear = friction force × distance through which it is moved.

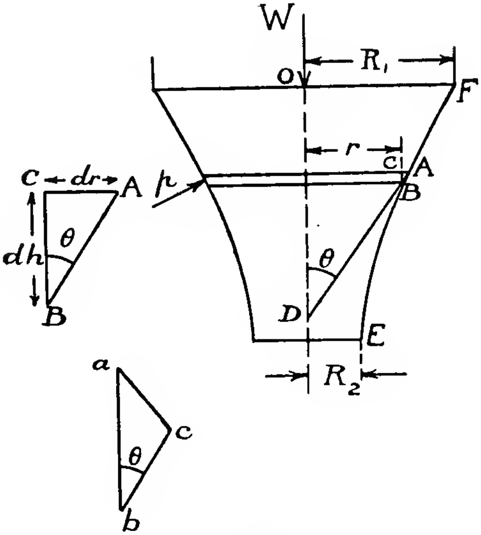

Fig. 80.

Now ab is the vertical wear and if ac is perpendicular to bc we have

$$\sin\theta = \frac{ac}{ab} = \frac{r}{\mathrm{AD}}$$

$$\therefore\ \mathrm{AD} = \frac{r.\,ab}{ac}$$

Now work done against friction per second

$$= \mu\, p\,.\,v = \mu p\, 2\pi r n$$

$\therefore$ ac is proportional to this, i.e.

$$ac = k.\mu p.2\pi\, r.n$$

where k = some constant and since, μ, p, and n are assumed constant we may write

$$ac = \mathrm{K}.\,r$$

$$\therefore\ \mathrm{AD} = \frac{r.\,ab}{\mathrm{K}r}$$

$$= \frac{ab}{\mathrm{K}}$$

$\therefore$ if the vertical wear ab of the pivot is constant the length AD is constant

The curve FAE therefore is such a curve that its tangent AD is of constant length.

This curve is called the tractrix [see p. 221].

§ 58. (E) **The Root Mean Square.**—It will be readily seen that the mean value of sin x between the limits $x = 0$ and $x = 2\pi$ is zero, for in the graph the area below the x-axis is equal to the area above it. In many cases a quantity has the same effect whether it is positive or negative, and in such cases one might add on the negative areas as if they were positive and find the mean as before. This would be convenient enough graphically, but would make the integration a little trying. In some cases, however, it is the mean of the square of a quantity that is important, and here no question of negative values arises, since a square is essentially positive.

The heat generated by an electric current is proportional to the square of the current.

Heat generated per second = $\mathrm{C}^2\mathrm{R}$,

where C = current

R = resistance.

If C varies the heat generated in time t is

$$\mathrm{W} = \int_0^t \mathrm{RC}^2 dt = \mathrm{R}\int_0^t \mathrm{C}^2 dt.$$

In this case it is the square of the current that we are concerned with, and if we knew *the mean value of* C^2,

$$\text{Mean } C^2 = K,$$

then the heat generated would be

$$RKt.$$

Suppose $\bar{C}$ is the constant current that generates the same amount of heat in the time t.

Then

$$W = R\bar{C}^2 t = R\int C^2 dt,$$

and

$$\bar{C} = \sqrt{\frac{1}{t}\int C^2 dt}$$

which is called the *root mean square current*, is what is important for us to know, especially in the case of *alternating* currents.

For example, suppose an alternating current to be given by

$$C = C_0 \sin \omega t,$$

then the root mean square over one complete period $\frac{2\pi}{\omega}$

$$\bar{C} = \sqrt{\left\{\frac{\omega}{2\pi} C_0^2 \int_0^{\frac{2\pi}{\omega}} \sin^2 \omega t dt\right\}}$$

$$= \sqrt{\left\{\frac{\omega}{2\pi} C_0^2 \cdot \frac{1}{\omega}\int_0^{\omega t = 2\pi} \sin^2 \omega t d\,(\omega t)\right\}}$$

$$= \frac{C_0}{\sqrt{2\pi}} \sqrt{\left\{\tfrac{1}{2}[\omega t - \cos \omega t \sin \omega t]_0^{\omega t = 2\pi}\right\}}$$

(see § 42, 5° (2).)

$$= C_0 \sqrt{\tfrac{1}{2}\frac{(2\pi)}{2\pi}}$$

$$= \frac{1}{\sqrt{2}} C_0.$$

GRAPHICAL METHOD.—Graphically the root mean square could be determined by constructing the graph of C^2 from a knowledge of C and then proceeding as before, but there is a better method.

Instead of determining the position of a point P (Fig. 81) by means of its rectangular co-ordinate (x, y)

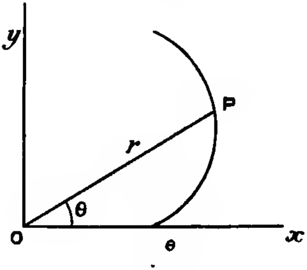

FIG. 81.

we may use the *polar* co-ordinates

$$r = \text{OP},\ \theta = x\hat{\text{O}}\text{P}.$$

A relation between r and θ will determine a curve as before, for example the curve (a part of it)

$$r = e^{\theta} \qquad \text{(A)}$$

is shown in Fig. 81. To trace this curve we measure along the radius OP corresponding to each angle θ, a length given by the equation (A).

THE "AREA OF THE CURVE," i.e. *of the sector* OAP (Fig. 82) is now considered as the sum of the triangles OAP_1, OP_1P_2, OP_2P_3, . . . approximately.

Now $\text{OP} = r$, $\text{OQ} = r + \delta r$, $\text{POQ} = \delta\theta$; so the area of the triangle

$$\text{OPQ} = \tfrac{1}{2}\,\text{OP}\,.\,\text{OQ}\sin\text{POQ}$$

(see Introduction.)

$$= \tfrac{1}{2}\,r\,.\,(r + \delta r)\sin\delta\theta$$
$$= \tfrac{1}{2}\,r\,.\,r\,.\,\delta\theta \text{ (approx.)}$$
$$= \tfrac{1}{2}\,r^2\delta\theta.$$

Taking this as the approximate area of an elementary sector POQ, we get area of whole sector

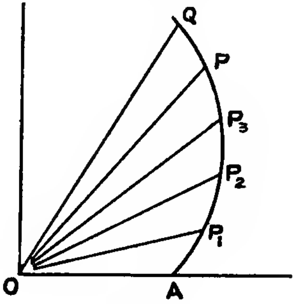

Fig. 82.

$$\text{AOP} = \int_0^\theta \tfrac{1}{2}\,r^2 d\theta$$
$$= \tfrac{1}{2}\int_0^\theta r^2 d\theta.$$

For the curve $r = e^\theta$, area of sector

$$= \tfrac{1}{2}\int_0^\theta r^2 d\theta = \tfrac{1}{2}\int_0^\theta e^{2\theta} d\theta = \tfrac{1}{4}\,(e^{2\theta} - 1).$$

To find the root mean square of C, suppose we now draw a curve (Fig. 82*a*), taking t as the angle AOP (in radians) and C as the radius OP.

Then the area AOP $= \tfrac{1}{2}\int_0^t \text{C}^2 dt.$

Hence *the root mean square = the square root of twice the area AOP divided by t.*

§ 59. (C) **Problems on the Flow of Water.**—A number of problems upon the flow of water introduce the operation of integration.

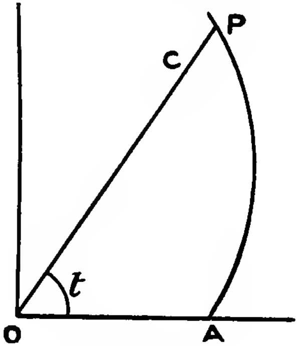

Fig. 82*a*.

Flow over a Rectangular Notch or Weir.—Consider an elementary strip of water at depth h (Fig. 83) below the surface. This will possess a velocity due to falling a height h,

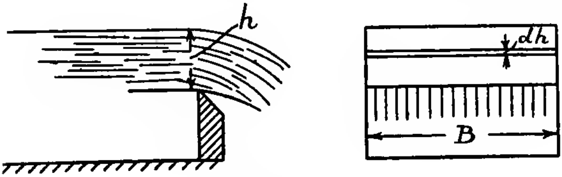

Fig. 83.

$$\text{i.e. } v = \sqrt{2gh}.$$

∴ flow of water through element per second = C_d × area of strip × velocity, where C_d is a constant called the *coefficient of discharge*

∴ elementary flow

$$= dQ = C_d \times \text{area of strip} \times \text{velocity}$$
$$= C_d \,.\, B dh \sqrt{2gh}$$
$$= \sqrt{2g} \,.\, C_d \,.\, B h^{\frac{1}{2}} dh$$
$$\therefore \text{total flow} = Q = \int_0^H \sqrt{2g} \,.\, C_d B h^{\frac{1}{2}} dh$$
$$= \sqrt{2g} \,.\, C_d B \int_0^H h^{\frac{1}{2}} dh$$
$$= \tfrac{2}{3} \sqrt{2g}\, C_d B \Big[h^{\frac{3}{2}} \Big]_0^H$$
$$= \tfrac{2}{3}\, C_d \,.\, BH \sqrt{2gH}.$$

TRIANGULAR NOTCH OR WEIR.—In this case (Fig. 84) we have as before

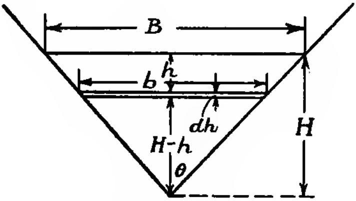

FIG. 84.

$$dQ = \sqrt{2g} \,.\, C_d \,.\, b h^{\frac{1}{2}} dh.$$

Now $b = 2(H - h) \tan \theta$

$$\therefore\ dQ = \sqrt{2g} \,.\, C_d \,.\, 2 \tan \theta\, (H - h) h^{\frac{1}{2}} dh$$
$$\therefore\ Q = \sqrt{2g} \,.\, C_d \,.\, 2 \tan \theta \int_0^H (H h^{\frac{1}{2}} - h^{\frac{3}{2}}) dh$$
$$= \sqrt{2g}\, C_d \,.\, 2 \tan \theta \,.\, \Big[\tfrac{2}{3} H h^{\frac{3}{2}} - \tfrac{2}{5} h^{\frac{5}{2}} \Big]_0^H$$
$$= \sqrt{2g} \,.\, C_d \,.\, 2 \tan \theta \,.\, \frac{4}{15} H^{\frac{5}{2}}$$
$$= \frac{8}{15} C_d \tan \theta\, H^2 \sqrt{2gH}$$
$$\text{or} = \frac{4}{15} C_d\, BH \sqrt{2gH} \quad . \quad . \quad . \quad (2)$$

TIME TO EMPTY A RECTANGULAR TANK WITH A

SMALL ORIFICE.—Let the area of the orifice be a (Fig. 85) and that of the tank [in plan] be A.

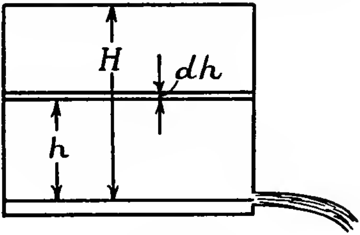

FIG. 85.

Let the head of water at any instant be h and let its surface descend through a distance dh in an element of time dt. Then velocity of flow through orifice.

$$= v = \sqrt{2gh}$$

$\therefore$ flow through orifice in time dt

$$= C_d \times \text{area of orifice} \times \text{velocity} \times dt$$

$$= C_d \,.\, a \sqrt{2gh} \,.\, dt$$

but the flow through the orifice must be equal to the diminution of volume in the tank, i.e. Adh

$$\therefore \ A dh = C_d \, a \sqrt{2gh} \, dt$$

$$\therefore \ dt = \frac{A dh}{\sqrt{2g} \,.\, C_d \,.\, a \ h^{\frac{1}{2}}}$$

. . time required to empty tank = t

$$= \int_H^O \frac{A}{\sqrt{2g}\, C_d \, a} \,.\, \frac{dh}{h^{\frac{1}{2}}}$$

$$= \frac{A}{\sqrt{2g} \,.\, C_d \,.\, a} \int_H^O h^{-\frac{1}{2}} \, dh$$

$$= \frac{A}{\sqrt{2g} \,.\, C_d \,.\, a} \left[2h^{-\frac{1}{2}+1} \right]_H^O$$

$$= \frac{2A\sqrt{H}}{\sqrt{2g} \,.\, C_d \,.\, a} = \frac{A}{aC_d}\sqrt{\frac{2H}{g}} \quad . \quad . \quad (3)$$

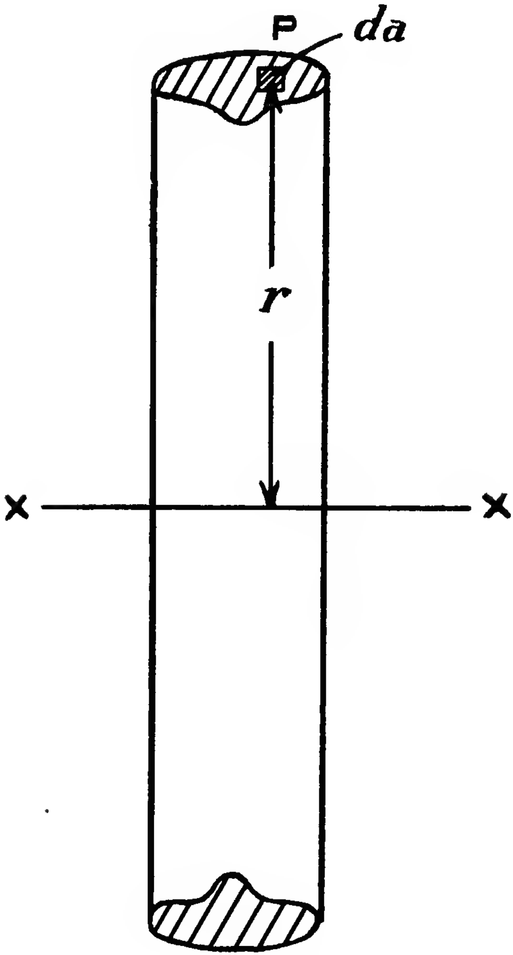

FIG. 86.

If the tank is of a different shape the method will be the same but the area A will not be a constant, and so the integration will be rather more complicated.

§ 60. (M) **Kinetic Energy Stored in a Flywheel.**—Consider an element *da* of area situated at a point P (Fig. 86) in the cross section of a flywheel rotating at *n* revolutions per second.

Then if ρ is the density of the material [weight of unit volume] the mass of the ring of which *da* is the section will be equal to

$$dM = \frac{\text{weight of ring}}{g} = \frac{\rho \,.\, 2\pi r da}{g}$$

and it has a velocity $2\pi rn$

$\therefore$ kinetic energy of element $= dE = \frac{1}{2}$ mass × velocity^2

$$= \tfrac{1}{2}\, dM \,.\, (2\pi rn)^2$$
$$= 2\pi^2 n^2 r^2 dM$$

$\therefore$ total K. E. of flywheel $= E = 2\pi^2 n^2 \int r^2 dM$ but $\int r^2 dM$ is what we have called the *polar moment of inertia* of the flywheel $= J$

$$\therefore \text{K. E.} = 2\pi^2 n^2 J \quad . \quad . \quad . \quad . \quad (1)$$

It is common to express this in terms of ω, the angular velocity which is equal to $2\pi n$

$$\text{then K. E.} = \tfrac{1}{2} J\omega^2 \quad . \quad . \quad . \quad (2)$$

If k is the polar radius of gyration of the flywheel,

$$J = Mk^2 = \frac{Wk^2}{g}$$

$$\text{then K. E.} = \tfrac{1}{2} M\omega^2 k^2 = \frac{\frac{1}{2} W\omega^2 k^2}{g} \quad . \quad (3)$$

§ 61. (C) **The Deflections of Beams.**—It can be shown that in a beam the relation holds that the slope θ of the beam at any point is given when such slope is small by the relation

$$\tan\theta = \int \frac{\mathrm{M}dx}{\mathrm{EI}} \quad . \quad . \quad . \quad (1)$$

where M = the bending moment at any point
E = Young's Modulus
I = M. I. of cross section about neutral axis.

and the deflection y of the beam is given by

$$y = \int \tan\theta \, dx \quad . \quad . \quad .. \quad (2)$$

To get the deflection y therefore from the bending moment we have to integrate twice; this is written

$$y = \iint \frac{\mathrm{M}dx^2}{\mathrm{EI}} \quad . \quad . \quad . \quad (3)$$

As a simple example of this take the case of a beam of span l simply supported at the ends and carrying a uniformly distributed load of p units per unit length (Fig. 87).

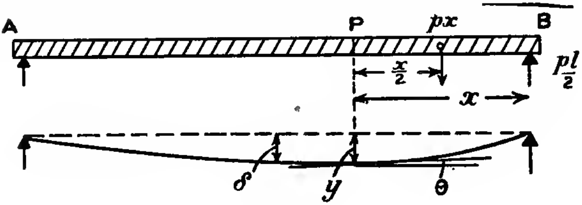

FIG. 87.

Consider any point P along the beam.

The forces acting to the right are an upward reaction equal to $\frac{pl}{2}$ at the end B and a downward force px acting at the centre of the length PB.

∴ moment of forces to the right of B

$$= \text{bending moment at P} = \text{M} = \frac{pl\,.\,x}{2} - \frac{px\,.\,x}{2}$$

$$= \frac{p}{2}(lx - x^2) \quad . \quad . \quad . \quad (4)$$

$$\therefore \text{ from (1) } \tan\theta = \int_0^x \frac{\frac{p}{2}(lx - x^2)\,dx}{\text{EI}}$$

∴ if E and I are constant

$$\text{EI}\tan\theta = \frac{p}{2}\left(\frac{lx^2}{2} - \frac{x^3}{3}\right) + \text{C}_1 \quad . \quad . \quad . \quad (5)$$

Now from symmetry the slope of the beam is zero at the centre where $x = \frac{l}{2}$

$$\therefore 0 = \frac{p}{2}\left(\frac{l\,.\,l^2}{8} - \frac{l^3}{24}\right) + \text{C}_1$$

$$\text{i.e. C}_1 = -\frac{pl^3}{16}(1 - \tfrac{1}{3}) = -\frac{pl^3}{24}$$

$$\therefore \text{EI}\tan\theta = \frac{plx^2}{4} - \frac{px^3}{6} - \frac{pl^3}{24} \quad . \quad (6)$$

∴ from equation (2)

$$\text{EI}\,.\,y = \int \text{EI}\tan\theta\,.\,dx$$

$$= p\int\left(\frac{lx^2}{4} - \frac{x^3}{6} - \frac{l^3}{24}\right)dx$$

$$= p\left(\frac{lx^3}{12} - \frac{x^4}{24} - \frac{l^3x}{24}\right) + \text{C}_2 \quad . \quad . \quad (7)$$

Now $y = 0$ at the end where $x = 0$ $\therefore \text{C}_2 = 0$

$$\therefore \text{EI}\,.\,y = \frac{px}{12}\left(lx^2 - \frac{x^3}{2} - \frac{l^3}{2}\right) \quad . \quad . \quad (8)$$

The maximum deflection δ occurs at the centre where $x = \frac{l}{2}$

$$\therefore \text{EI} \,.\, \delta = \frac{pl}{24}\left(\frac{l^3}{4} - \frac{l^3}{16} - \frac{l^3}{2}\right)$$

$$= \frac{pl^4}{2 \times 24}\left(\tfrac{1}{2} - \tfrac{1}{8} - 1\right)$$

$$= \frac{pl^4}{2 \times 24}\left(-\tfrac{5}{8}\right)$$

$$= \frac{-5pl^4}{384}$$

$$\therefore \delta = \frac{-5pl^4}{384\,\text{EI}} \quad . \quad . \quad . \quad (9)$$

The minus sign indicates that the deflection is downwards.

The deflections of beams loaded and supported in various other manners are obtained in a similar way and will be found in the various text-books dealing with the subject.

EXERCISES 7.

1. Show that the centroid of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is at distances $\frac{4a}{3\pi}$ and $\frac{4b}{3\pi}$ from the axes.

2. A triangle has its vertex at the surface of a liquid and has its base horizontal. If h is the height of the triangle, find the depth of its centre of pressure.

3. The pressure of one pound of saturated steam at 347° F. is 130 lb. per sq. in., and the volume is 3·44 cub. ft. Find the work done in an adiabatic expansion which doubles the volume, $\gamma = 1{\cdot}135$. (M.)

4. Find the work done in suddenly compressing one pound of dry air originally at 32° F. to $\frac{3}{4}$ths of its original volume. Use the equations $\text{PV} = 53{\cdot}18\,\text{T}$ (abs) and $\text{PV}^{1\cdot404} = \text{constant}$.

P = pressure in lbs. per sq. ft.
V = Volume in cu. ft.

Thus get the temperature after expansion and apply the formula: work done $= \frac{53{\cdot}18\ (T_1 - T_2)}{{\cdot}404}$. (M.)

5. A certain volume of air is drawn into a cylinder at a pressure 120 lb. per sq. in., and is expanded isothermally until the volume is 6 times as great. Find the mean pressure during inlet and expansion. (M.)

6. Find the mean pressure under the conditions of the previous question if the expansion is adiabatic taking $\gamma = 1{\cdot}408$. (M.)

7. Find the area included between the axis of x and the curve $9y = x^2\ (x + 3)$, x and y being in inches.

8. Find the co-ordinates of the centroid of the area included between the curve $y = \frac{a^2}{a^2 + x^2}$ and the axis of x.

9. Find the centroid of the area included between one semi-undulation of the curve $y = 6 \sin \frac{x}{2}$ and the axis of x.

10. Find the centroid of the circular spandril formed by the quadrant of a circle of radius r and the tangents at its extremities.

11. A trapezium of depth h has one side a of its parallel sides a, b, in the surface of a liquid. Prove that the depth of the centre of pressure is

$$\frac{a + 3\,b}{2\,(a + 2\,b)} \cdot h.$$

12. Find the radius of gyration of a uniform circular disk of radius r about an axis normal to its plane passing through a point on its circumference. (C.)

13. A bending moment diagram of a beam of span l is made up of a triangle of height $\frac{pl^2}{16}$ with apex at

the centre and a parabola, extending from the right-hand end to the centre, of height $\frac{pl^2}{32}$. Find the position of the centroid of the diagram. (C.)

14. The moment of inertia of a channel section about its base is 13·639 inch units, and the centroid is 1·185 inches from the base. If the area of the section is 5·219 sq. in., find the moment of inertia and radius of gyration about a line through the centroid parallel to the base. (C.)

15. If the root mean square value of an alternating electric current is ten amperes, what is the maximum value? (E.)

16. What will be the *average* value of the current in one semi-alternation in the previous exercise? (E.)

17. Find by means of the theorem of Pappus the distance from the diameter of the centroid of (*a*) the surface of a hemisphere, (*b*) a solid hemisphere.

18. A rectangular beam 12 ft. long, 1 ft. deep and 5 in. wide rests on supports at its ends. What uniformly distributed load will cause a deflection equal to $\frac{1}{1000}$ of its span. $E = 1{\cdot}8 \times 10^6$ lb. per sq. in. (C.)

19. A cast-iron pipe of 18 in. internal diameter and 1 in. thick, rests on supports 40 feet apart. Find the maximum bending stress at the centre when the pipe is full of water. Take weight of cast iron = 450 lb. per cub. ft. and of water 62·3 lb. per cub. ft. (C.)

20. The resilience or work done in bending of a beam is given by $\int_0^L \frac{M^2 dx}{2EI}$ where E and I are constant. If $M = \frac{w}{2}(Lx - x^2)$ as is the case for uniformly distributed loading, find the resilience. (C.)

21. The entropy ϕ of a substance is defined by $\phi = \int \frac{dQ}{T}$ where Q is the heat absorbed and T is the

absolute temperature. If the specific heat of water is taken as 1, the entropy above freezing point is $\int_{493}^{T_1}\frac{dT}{T}$; find the entropy of one pound of water at 300° F. (761° F. absolute). (M.)

22. If the specific heat of superheated steam is ·48, find the entropy of a pound of steam at 300° F superheated to 390° F. The latent heat L at this temperature (300° F.) may be taken as 903 so that the formula for the entropy becomes

$$\phi = \int_{493}^{761}\frac{dT}{T} + \frac{L}{T} + \cdot 48\int_{761}^{851}\frac{dT}{T}$$

It is $\frac{L}{T}$ during the evaporation period, because T is constant and $\int dQ = L$. (M.)

23. A vertical shaft 4 inches in diameter turns on a flat pivot. The weight of the shaft with its various fittings is 2500 lb. If $\mu = \cdot 08$ and the speed is 140 revolutions per min., find the horse-power used in friction by the pivot. (E.) (M.)

24. In the above question find the horse-power lost if the pivot were a cone frustum, the cone angle being 60° and the least diameter 1½ inches (E, M.)

25. Will the work absorbed in friction in a Schiele pivot be less than or more than that of a truncated cone of the same outer radius, the half angle of cone being 45° and the angle of the tangent to tractrix at the longer radius being 45°? The radius of the small end of the conical pivot may be taken as one-half the large radius. (E, M.)

26. Two tanks whose surface areas A are equal to each other have a difference in water level of H and are connected by a small orifice O below the level of the water in either and of area a. If C_d be the coefficient of discharge, find the time from the opening of the orifice until the level in the two tanks is equal (note that the change in head is twice the rise or fall). (C.)

27. A large rectangular orifice of breadth b and

depth h is placed in the side of a tank. If H is the head of water above the top of the orifice show that the flow is $Q = \frac{2}{3}bC_d\sqrt{2g}\,\{(H+h)^{3} - H^{\frac{3}{2}}\}$.

28. Find the time to empty a rectangular chamber 120 ft. square containing 15 ft. depth of water, which is allowed to flow out through a rectangular orifice 2 ft. by 1 ft., the top of which is level with the floor of the chamber, take $C_d = \cdot 62$. (C.)

29. A tank 10 ft. square and 10 ft. deep has a circular orifice 4 inches in diameter in the bottom. If the tank is filled and the water turned off, how long will it take to empty taking $C_d = \cdot 62$? (C.)

30. A hemispherical cistern is 20 ft. in diameter, and it is full of water. How many minutes will it take to lower the depth of the water 5 ft., if the water escapes through a 3 inch diameter sharp-edged hole in the bottom of the cistern. Take the coefficient of discharge as ·60. (C.)

31. Calculate the discharge through a 90° angular notch, the height of the still water surface above the bottom being 15·5 inches; take ·60 as the coefficient. (C.)

32. A tube of length $2r$, closed at both ends and full of liquid, revolves in a horizontal plane about an axis, at right angles to and bisecting its own axis. Prove that the intensity of pressure on its end will be $\frac{w\omega^2r^2}{2g}$ where w is the weight per unit volume of the liquid and ω is the angular velocity. (C.)

33. A cantilever is fixed horizontally at one end and is free at the other end, the length being l. Assuming that the deflections y are given by $\frac{d^2y}{dx^2} = \frac{M}{EI}$ and M at a distance x from the free end is Wx, find the deflection at the free end. (C.)

34. In arch problems the quantity $\int y^2dx$ is used in determining the thrusts. For a parabolic arch $y = \frac{4r}{L^2}(Lx - x^2)$; find $\int_0^L y^2dx$ for such an arch. (C.)

35. The following integration also occurs in problems on parabolic arches $\int_{a\mathrm{L}}^{\mathrm{L}} (\mathrm{L}^2 x - 2\mathrm{L}x^2 + x^3)\,dx$. Evaluate it. (C.)

ANSWERS TO EXERCISES.

Exercises 7.

2. $\frac{3h}{4}$. **3.** 42200 ft.-lb. **4.** 7980 ft.-lb. **5.** 55·8 lb. per sq. in. **6.** 45·4 lb. per sq. in. **7.** ·75 sq. in. **8.** $0, \frac{a}{4}$. **9.** $\frac{3\pi}{4}$. **10.** ·223 r. **12.** $\frac{\sqrt{6}\,r}{2}$. **13.** $\frac{7\,l}{16}$ from the right-hand end. **14.** I = 6·316 inch units, k = 1·01 in. **15.** 14·14 amperes. **16.** 9·01 amperes. **17.** (a) Surface $= 4\pi r^2 =$ arc × path of centroid $= \pi r \times 2\pi x \therefore = \frac{2r}{\pi}$, ($b$) $\frac{4r}{3\pi}$. **18.** 4800 lb. **19.** 2640 lb. per sq. in. **20.** $\frac{w^2\mathrm{L}^5}{240\ \mathrm{E\,I}}$. **21.** $\log_e \frac{761}{493} = \cdot 432$. **22.** $\log_e \frac{761}{493} + \frac{903}{761} + \cdot 48 \log_e \frac{851}{761} = 1\cdot 67$. **23.** ·59. **24.** 1·32. **25.** Schiele $= \mu \mathrm{WR}_1 \sqrt{2}$ and conical $\frac{\mu \mathrm{WR}_1 7 \sqrt{2}}{9}$ which is less. **26.** $t = \frac{\mathrm{A}\,\sqrt{\mathrm{H}}}{\mathrm{C}da\,\sqrt{2g}}$. **28.** 3 hrs. 7 min. nearly. **29.** 25 min. nearly. **30.** 37 min. 19 sec. **31.** 4·87 cub. ft. per second. **33.** $\frac{\mathrm{W}l^3}{\mathrm{EI}3}$. **34.** $\frac{8\,r^2\mathrm{L}}{15}$. **35.** $\frac{\mathrm{L}^4\,(1 - a)\,(1 + a - 5\,a^2 + 3\,a^3)}{12}$.

CHAPTER VIII.

PARTIAL DIFFERENTIATION.

§ 62. **Several Independent Variables.**—A physical quantity may depend upon several other quantities which are independent of each other, instead of depending upon one only. For instance, the volume v of a given mass of a perfect gas is connected with the pressure p and the (absolute) temperature θ in the following way:—

$$pv = \mathrm{R}\theta, \quad . \quad . \quad . \quad (1)$$

where R is a constant. So

$$v = \mathrm{R}\frac{\theta}{p} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The temperature and pressure may here be adjusted independently of each other. We might vary the temperature θ alone, keeping the pressure constant; then the rate of change of volume with temperature would be got by differentiating equation (2),

$$\frac{\partial v}{\partial \theta} = \frac{\mathrm{R}}{p}.$$

Or again we might keep θ constant and vary the pressure; the rate of change of the volume with the pressure would be

$$\frac{\partial v}{\partial p} = -\frac{\mathrm{R}\theta}{p^2}.$$

When a quantity z depends upon two quantities x and y, then the rate of change of z with respect to x when y is supposed to remain constant is called a partial differential coefficient. It is written $\frac{\partial z}{\partial x}$ (pronounced *day z* over *day x*).

To take another example, suppose

$$z = 2x^2 + 3xy + y^2 + 2x + 1$$

then

$$\frac{\partial z}{\partial x} = 4x + 3y + 2$$

$$\frac{\partial z}{\partial y} = 3x + 2y.$$

Or again let

$$z = \tan^{-1}\frac{y}{x}$$

then

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}} \times -\frac{y}{x^2} = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}} \times \frac{1}{x} = \frac{+x}{x^2 + y^2}.$$

HIGHER DIFFERENTIAL COEFFICIENTS.—We can as before find the slope of the slope with respect to x, i.e. for the first of the above examples

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}(4x + 3y + 2) = 4 \text{ (a constant).}$$

And similarly

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}(3x + 2y) = 2.$$

But we have in this case another second order slope, viz. the slope with regard to y of the slope with regard to x,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(4x + 3y + 2) = 3.$$

And also

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(3x + 2y) = 3.$$

It will be noted that in this case

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

This is always the case for ordinary laws. For our second example

$$z = \tan^{-1}\frac{y}{x}$$

we have

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2 + y^2}\right) = -y\frac{\partial}{(x^2 + y^2)}(x^2 + y^2)^{-1} \times \frac{\partial(x^2 + y^2)}{\partial x}$$

$$= -y \times (-1)(x^2 + y^2)^{-2} \times 2x = \frac{2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \frac{-2xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial x \, \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right) = \frac{(x^2 + y^2)\frac{\partial x}{\partial x} - x\frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 z}{\partial y \, \partial x} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

§ 63. **Differentials and Small Variations.**—SMALL VARIATIONS.—To return to one of our earliest examples, suppose x represents the distance covered by a motor-car at a given instant and t represents the time taken in covering this distance. The speed is $\frac{dx}{dt}$: so the distance covered in a small interval of time following δt will be

$$\frac{dx}{dt} \times \delta t$$

approximately; this is not exact because $\frac{dx}{dt}$ is the speed at the beginning of the interval δt, and the speed is varying, but if δt is very small the variation in the speed will likewise be small, and hence the error will be negligible.

To be more precise, let us suppose that the car is moving with a speed of 40′ per second and has a uniform acceleration of 1′ per second per second; let the interval of time $\delta t = \frac{1}{10}$ second. Then the quantity which approximates to the distance covered

$$\frac{dx}{dt}\,\delta t = 40' \times \tfrac{1}{10} = 4',$$

a "small quantity". But the average speed during the interval is $40\frac{1}{20}'$ per second, and the exact distance covered is

$$40\tfrac{1}{20} \times \tfrac{1}{10} = 4\tfrac{1}{200}'.$$

The error made is therefore $\frac{1}{200}'$, i.e. only $\frac{1}{800}$ of 4′, and this we call a "small quantity of second order". The proportional error is still less if we take a shorter interval, say $\frac{1}{100}$ second. In this case

$$\frac{dx}{dt}\,\delta t = 40' \times \tfrac{1}{100} = 0{\cdot}4'.$$

The average speed is $40\frac{1}{200}$, and the exact distance covered is $(0{\cdot}4' + \frac{1}{20000}')$. The error $\frac{1}{20000}'$ is now only $\frac{1}{8000}$ of 0·4′.

Thus in general the variation of any quantity y in terms of x may be taken as

$$\delta y = \frac{dy}{dx}\,\delta x$$

when δx is small, and the smaller δx is the closer is the approximation.

These small variations

$$\delta x = \text{variation in } x$$

$$\frac{dy}{dx}\,\delta x = \text{variation in } y \text{ corresponding,}$$

are generally called *differentials.*

Thus if

$$y = x^3$$

the differential of y is

$$\delta y = 3x^2\delta x.$$

We obtain small variations similarly when the dependent quantity varies with several others. The relation between pressure, volume and temperature for a given mass of perfect gas is

$$v = \mathrm{R}\,\frac{\theta}{p} \text{ (see § 62).}$$

A variation in v when θ varies, p being constant, is

$$\delta_p v = \frac{\mathrm{R}}{p}\,\delta\theta\,;$$

and a variation in v when p varies, θ being constant is

$$\delta_\theta v = -\,\frac{\mathrm{R}\theta}{p^2}\,\delta p.$$

TOTAL VARIATION.—What will be the variation in v in the last example when *both θ and p vary?* The answer is that instead of supposing θ and p to vary simultaneously, we can suppose that θ varies first to its new value $\theta + \delta\theta$, p being constant, and that p then varies to *its* new value $p + \delta p$, θ remaining constant at $\theta + \delta\theta$. Then the sum of these two variations is the total variation in v. The first variation in v is

$$\delta_p v = \frac{R}{p}\,\delta\theta.$$

The second variation in v, i.e. with the temperature at $\theta + \delta\theta$, will be very nearly what it would be at θ (because $\delta\theta$ is small). Thus

$$\delta_\theta v = -\frac{R(\theta+\delta\theta)}{p^2}\,\delta p \doteq -\frac{R\theta}{p^2}\,\delta p.$$

Finally the total variation is

$$\begin{aligned}\delta v &= \delta_p v + \delta_\theta v\\ &= \frac{\partial v}{\partial\theta}\,\delta\theta + \frac{\partial v}{\partial p}\,\delta p\\ &= \frac{R}{p}\,\delta\theta - \frac{R\theta}{p^2}\,\delta p,\end{aligned}$$

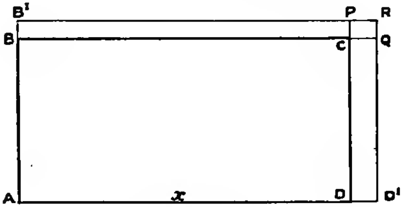

FIG. 88.

very nearly, neglecting small quantities of second order.

In general if z depends upon x and y,

$$\delta z = \frac{\partial z}{\partial x}\,\delta x + \frac{\partial z}{\partial y}\,\delta y.$$

Let us take a simple case to see more clearly the kind of error we are neglecting. The area of a rectangle whose sides are x and y is

$$z = xy$$

Then according to our work,

$$\delta z = \frac{\partial z}{\partial x}\,\delta x + \frac{\partial z}{\partial y}\,\delta y = y\delta x + x\delta y.$$

Now δz is really the increase in the area of the rectangle when the side AD increases by DD′ $= \delta x$, and the side AB increases by BB′ $= \delta y$. The quantity $y\delta x$ is the area DD′QC, while $x\delta y$ is BCPB′. So the quantity neglected is here the area CQRP, and this is small compared with the total increase in z.

EXERCISES 8.

1. If $u = x^n y^m$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

2. If $z = (x + y)^2$, show that $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$.

3. If $r = \sqrt{x^2 + y^2}$, find $\frac{\partial r}{\partial x}$ and $\frac{\partial r}{\partial y}$.

4. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0$ is the general equation to the series of curves known as "conics" find the slope of the tangent $\left(\text{i.e. } \frac{dy}{dx}\right)$.

5. Calculate the difference for one minute in a table of $\log_{10}$ sines in the neighbourhood of 45°. Proceed as follows: $y = \log_{10}\sin x$, $\delta y = \delta x\,\frac{d\log_{10}\sin x}{dx}$

$$= \frac{\pi}{60 \times 180}\,\frac{d\log_{10}\sin x}{dx} = \frac{\pi}{2{\cdot}3 \times 60 \times 180}\,\frac{d\log_e \sin x}{dx}.$$

6. The height of a tower is calculated by measuring a length of 100 ft. from the base and measuring the angle of elevation which comes to 30°. If the error in the angle be one minute, find the error in the computed height.

7. In a tangent galvanometer the current is proportional to the tangent of the inclination of the

needle. Find the portion of the scale where the proportional error in the current due to a given error of reading is least.

8. The height of a cliff is calculated by finding that a stone takes 3 seconds to reach the bottom. If the error in timing is $\frac{1}{10}$ second, what will be the error in the calculated height?

9. If w is the weight of a body in air and w' is its weight immersed in water, the specific gravity of the body is given by $\frac{w}{w - w'}$. If the weight in water is correct and that in air has a small error e, what will be the resulting error in the calculated specific gravity?

10. Take the same problem as before but take each measurement as liable to errors e_a and e_w respectively. If $x = 120{\cdot}5$ grm. and $y = 110{\cdot}9$ grm., $e_a = 3$ mg. and $e_w = 4$ mg. find the total possible percentage error in the calculation.

ANSWERS TO EXERCISES.

Exercises 8.

1. $ny^m x^{n-1}, mx^n y^{m-1}$. 3. $\frac{x}{r}, \frac{y}{r}$. 4. $-\frac{ax+hy+g}{hx+by+f}$.

5. ·000126. 6. ·47 ft. 7. 45°. 8. about 10 ft.

9. $\frac{ey}{(x-y)^2}$. 10. ·07 per cent.

CHAPTER IX.

DIFFERENTIAL EQUATIONS.

§ 64. **Some Examples.**—A differential equation is a relation between a variable quantity and one or more of its differential coefficients.

1°. The current C in a circuit whose resistance is R and whose inductance is L satisfies the relation

$$\mathrm{RC} + \mathrm{L}\frac{d\mathrm{C}}{dt} = \mathrm{V}$$

where V is the potential difference at the two extremities.

This is a differential equation of *first order*, since the first differential coefficient $\frac{d\mathrm{C}}{dt}$ only occurs in it. The independent quantity is the time t. The potential V we may suppose known and it then has an expression in terms of t. The coefficients R and L are simply constants.

2°. The acceleration of a body falling freely is given by

$$\frac{d^2x}{dt^2} = g = 32 \text{ in ft.-sec. units}$$

where x is the distance through which the body has fallen in time t. This is a differential equation of *second order*. It will be noted that neither x nor $\frac{dx}{dt}$ occur in this equation.

3°. Suppose a point moves in a straight line so that it always has an acceleration towards a point in this line proportional to its distance from the point. Then its acceleration

$$\frac{d^2x}{dt^2} = -kx$$

where k is a constant.

4°. To take a more general case, suppose the particle is urged towards the point with another given periodic force $= mb \sin pt$ (period $= 2\pi/p$, mass $= m$), and that there is also a resistance proportional to the velocity $\left(\text{to } \frac{dx}{dt}\right)$. Then the mass × acceleration = acting force;

$$m\frac{d^2x}{dt^2} = -m\mu x + mb \sin pt - mk\frac{dx}{dt}$$

or

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \mu x = b \sin pt.$$

If the right-hand side is absent, this is the relation satisfied by a *damped harmonic vibration*. With the right-hand side it is a *damped forced harmonic vibration*.

To *solve a differential equation* means to find an expression for the dependent quantity which satisfies it. We do not propose to give a systematic discussion of methods, but we will give the solutions for one or two important cases.

In the same way as in the case of integration, there is no *direct* method of solving a differential equation, we can only solve it by reducing it to some standard form of which the solution can be recognized.

§ 65. **Solutions.**—The solution of a differential equation involves an integration. Take the equation

$$\frac{dx}{dt} = k\, t^2.$$

This expresses that the velocity of a particle is proportional to the square of the time, and we want to find an expression for the displacement.

Evidently

$$x = k \int t^2 dt = \tfrac{1}{3} k\, t^3 + \mathrm{C}.$$

This is the law for some value of C.

To determine this value we have an *initial condition*: for example, initially $x = 1'$. Putting $t = 0$, $x = 1$ we get

$$1 = 0 + \mathrm{C}$$

which determines C:

$$\mathrm{C} = 1.$$

Hence $$x = \tfrac{1}{3} k\, t^3 + 1.$$

1°. (E) Take next a simple case of 1° (§ 64), namely the case where $\mathrm{V} = 0$; this means that the E.M.F. is cut out of a circuit which is then left to itself.

$$\mathrm{RC} + \mathrm{L}\frac{d\mathrm{C}}{dt} = 0$$

or

$$\frac{d\mathrm{C}}{dt} = -\frac{\mathrm{R}}{\mathrm{L}}\mathrm{C}.$$

We recognize at once that the law is exponential,[1] and as a matter of fact it is

$$\mathrm{C} = \mathrm{C}_0 e^{-\frac{\mathrm{R}}{\mathrm{L}}t}$$

as can be readily verified by substituting, for

---

[1] See p. 63.

$$\frac{dC}{dt} = C_0 \times \left(-\frac{R}{L}\right) \times e^{-\frac{R}{L}t}$$

$$= -\frac{R}{L} \cdot C_0 e^{-\frac{R}{L}t} = -\frac{R}{L} C.$$

Here C_0 is the value of C when $t = 0$.

When V is a constant, the solution of the equation quoted above is

$$C = Ke^{-\frac{R}{L}t} + \frac{V}{R}$$

where K is some constant. [Verify this by substitution.] For example if an E.M.F. equal to V is suddenly applied to a circuit in which there is initially no current, then $C = 0$ when $t = 0$; hence

$$0 = K \times 1 + \frac{V}{R} \text{ or } K = -\frac{V}{R}$$

and the formula for C is

$$C = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

Note that the second term $e^{-\frac{R}{L}t}$ soon becomes very small, so that C quickly approaches its limiting value, viz. $\frac{V}{R}$.

2°. We take next the equation

$$\frac{d^2x}{dt^2} = g.$$

At once, on integrating,

$$\frac{dx}{dt} = gt + A,$$

where A is a constant. Suppose initially the velocity $\frac{dx}{dt}$ (downwards) $= u$; then

$$u = 0 + A \text{ or } A = u$$

and
$$\frac{dx}{dt} = gt + u.$$

Integrate again

$$x = g \cdot \tfrac{1}{2}t^2 + ut + \mathrm{B},$$

where B is a constant. Suppose that we measure distances from the starting-point of the body; then

$$x = 0 \text{ when } t = 0,$$

so that

$$0 = 0 + 0 + \mathrm{B}, \quad \mathrm{B} = 0,$$

and we get finally the familiar formula for the displacement of a body moving with a constant acceleration g,

$$x = ut + \tfrac{1}{2}gt^2.$$

Simple Harmonic Motion.—3°. (M) We take next

$$\frac{d^2x}{dt^2} = -m^2x,$$

where we have written m^2 for k.

Note that

$$\frac{d \sin t}{dt} = \cos t$$

$$\text{and } \frac{d^2 \sin t}{dt^2} = -\sin t.$$

So that the second differential coefficient of $\sin t$ is $\sin t$ reversed in sign. This also applies to $\cos t$. This gives us a clue to a solution;

$$x = \sin mt$$

$$\text{or } x = \cos mt$$

would both do. If we combine them putting in constants as coefficients, we get

$$x = \mathrm{A} \sin mt + \mathrm{B} \cos mt \qquad (1)$$

and by substitution we can easily verify that this satisfies the equation.

Let us take a specific case.

A point moves with an acceleration μx towards a point 0, starting at rest from an initial distance d, where x is the distance at any time t. Then

$$\text{when } t = 0,\ x = d \text{ and } \frac{dx}{dt} = 0$$

(i.e. velocity is zero). Substitute in (1).

$$d = \text{A} \sin m.0 + \text{B} \cos m.0$$
$$\text{or } d = 0 + \text{B} \,.\, 1$$
$$\text{i.e. B} = d.$$

Now differentiate (1)

$$\frac{dx}{dt} = \text{A}m \cos mt - \text{B}m \sin mt.$$

Substituting the initial values

$$0 = \text{A}m \,.\, 1 - \text{B}m \,.\, 0$$
$$\text{whence A}m = 0 \text{ or A} = 0$$

and finally

$$x = d \cos mt$$
$$\frac{dx}{dt} = -\ dm \sin mt.$$

This is a case of simple harmonic motion (§§ 6, 16).

In this equation $x = d$ when $mt = 0$ and again equals d when $mt = 2\pi$

$$\therefore t = \frac{2\pi}{m}$$

is the time period between two successive identical positions or in the case of oscillation is the time of a double swing.

The most convenient form of this result is

$$t = 2\pi \sqrt{\frac{\text{Mass}}{\text{Force at unit distance}}}$$

and will be understood from the following :—

$$\frac{d^2x}{dt^2} = \text{acceleration } (a)$$

$$\therefore \text{when } x = -\ 1,\ a = +\ m^2$$

(we take $x = -1$, not $x = +1$, so that the acceleration towards the origin may be positive)

$$\therefore m = \sqrt{a}$$

$$\text{also } \frac{1}{a} = \frac{\text{mass}}{\text{force}}$$

$$\therefore \frac{1}{m} = \sqrt{\frac{\text{Mass}}{\text{Force to cause unit displacement}}}$$

Euler's Formula for Long Struts and Columns.—(4). (C) Although the strength of columns has no apparent connexion with simple harmonic motion, it is as a matter of fact derived from the same form of differential equation.

Referring to Fig. 89, suppose that a strut is so long that the direct stress when deflected is negligible compared with the stress due to bending.

Now suppose that for some reason or other the column becomes slightly deflected (the deflection is shown exaggerated in the figure); then the bending moment at any point Q will be equal to

$$M_Q = P \cdot y \quad \dots\dots\dots \quad (1)$$

For a beam with a small deflection we may write

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = -\frac{Py}{EI}$$

$$\text{i.e. } \frac{d^2y}{dx^2} = -\frac{P \cdot y}{EI}* \quad \dots\dots \quad (2)$$

Putting $\frac{P}{EI} = m^2$, this is the same general equation as that we have just considered and the general solution is

$$y = A \sin mx + B \cos mx \quad \dots\dots \quad (3)$$

* The sign is minus because a counter clockwise moment across the section, i.e. a positive moment, gives a negative value of $\frac{d^2y}{dx^2}$, i.e. causes the slope to decrease.

Now we see from the figure that y has the same value for equal positive and negative values of x, and

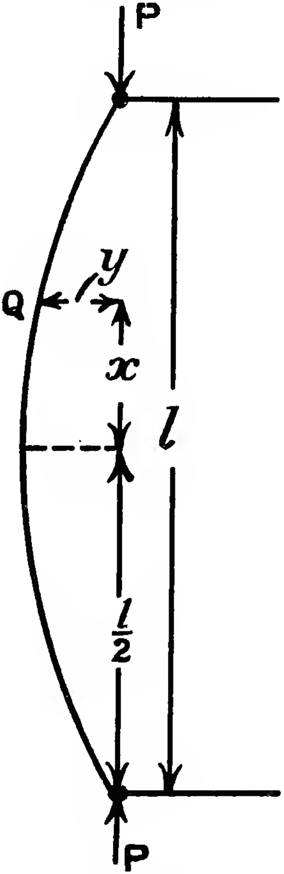

FIG. 89.

as sin $m\,x$ will become negative when x is negative while cos $m\,x$ is unchanged, A must be zero.

$$\therefore y = \mathrm{B} \cos m\,x.$$

Also when $x = \pm\frac{l}{2}$, $y = 0$.

$$\therefore\ 0 = \mathrm{B}\ \cos\frac{ml}{2} \qquad \ldots\ldots\ldots \quad (3)$$

The smallest angle whose cosine is 0 is 90° or $\frac{\pi}{2}$.

$$\therefore \frac{ml}{2} = \frac{\pi}{2}$$

$$\text{i.e. } m = \frac{\pi}{l}$$

$$\text{or } m^2 = \frac{\pi^2}{l^2}$$

$$\text{but } m = \frac{\mathrm{P}}{\mathrm{EI}}$$

$$\therefore \frac{\mathrm{P}}{\mathrm{EI}} = \frac{\pi^2}{l^2}$$

$$\text{or } \mathrm{P} = \frac{\pi^2\mathrm{EI}}{l^2}.$$

This is *Euler's formula.*

It gives us the load P which is theoretically required to maintain the column in an assumed deflected position, and it will be noted that this load is quite independent of the amount of the deflection and that if the deflection were doubled the load to keep it in such a deflected form would be the same. The load P therefore is called the *critical load upon the column* and with this load the column is in a state of unstable equilibrium.

If, therefore, the load upon the column is less than P and the column were given a slight deflection the column would straighten itself again, but if the load is ever so little in excess of P the deflection will go on increasing until the column fails. P thus may be regarded as the load which will cause the column to fail.

Equation (3) would also be satisfied by

$$m\frac{l}{2}=\frac{3\pi}{2}$$

$$\text{or } \frac{5\pi}{2} \ldots \text{ or any odd multiple of } \frac{\pi}{2},$$

but these solutions would be of no practical value because they would give larger values of P and the column would have buckled before such values were reached.

Differential Equations in General.—It is thought that the above simple examples will give an idea of the nature of differential equations. Further treatment is outside the scope of the present book and the general consideration of such equations forms one of the most difficult branches of advanced mathematics. The student who wishes to go further into the matter should consult Edwards' "Integral Calculus for Beginners," or Perry's "Calculus for Engineers". A more complete treatment will be found in Forsyth's "Differential Equations" [Macmillan].

Exercises 9.

1. Prove that the time of swing of a simple pendulum of length l which has a small swing is equal to $2\pi\sqrt{\frac{l}{g}}$.

[If θ is the angle of displacement,

$$\frac{d^2\theta}{dt^2} = -\frac{l}{g}\theta.]$$

2. A body is urged forward by a constant force A and there is a resistance proportional to its velocity.

Determine its equation of motion. [Proceed as follows :—

Resultant force $= F = A - bv$

$$\therefore \frac{W}{g} . a = A - bv$$

$$a = \frac{Ag}{W} - \frac{bg}{W} . v$$

$$= B - kv,$$

where B and k are constants.

$$\therefore a = \frac{dv}{dt} = B - kv.$$

Solve this for v and check your result by differentiation.]

3. Solve the equation $\frac{dy}{dx} + 2\frac{y}{x} = x^2$.

[Multiply by x^2

$$x^2 \frac{dy}{dx} + 2\,xy = x^4$$

$$\text{i.e. } x^2 \frac{dy}{dx} + y\frac{dx^2}{dx} = x^4$$

$$\frac{d\,(x^2y)}{dx} = x^4.$$

Now integrate.]

4. Solve the equation $\frac{xdy}{dx} + x + y = 0$

$$[\text{i.e. } x\frac{dy}{dx} + y = -x$$

$$\frac{d\,(xy)}{dx} = -x.$$

Now integrate.]

5. The equation of motion of two weights W and w connected by a string over a pulley is

$$(W + w)\frac{d^2s}{dt^2} = (W - w)\,g.$$

If $W = 3$ and $w = 2$ find the distance traversed after 1 second from starting.

ANSWERS TO EXERCISES.

9.

2. $v = \mathrm{C}e^{-kt} + \frac{\mathrm{B}}{k}$. **3.** $x^2y = \frac{x^5}{5} + \mathrm{C}$, i.e. $y = \frac{x^3}{5} + \frac{\mathrm{C}}{x^2}$. **4.** $x^2 + 2xy =$ constant. **5.** 3·2 ft.

CHAPTER X.

SOME FURTHER GEOMETRICAL APPLICATIONS AND SPECIAL CURVES.

§ 66. **Radius of Curvature.**—If QPS (Fig. 90) are three points on any curve and a circle be drawn through the points, the *circle of curvature* may be defined as the limiting position of this circle as the points Q and S both approach closer and closer to P.

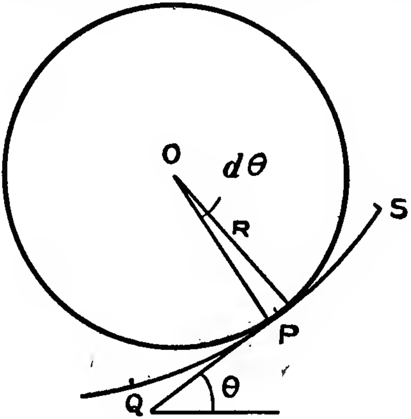

Fig. 90.

The centre O of this circle is called the centre of curvature, and since the radii O′O, S′O in Fig. 91 (which shows the limiting position of the points drawn to an enlarged scale) are at right angles (normal) to the circle at O′ and S′, and in the limit also normal to the curve, the centre of curvature may be defined as the intersection of two normals to the curve at points infinitely close to each other.

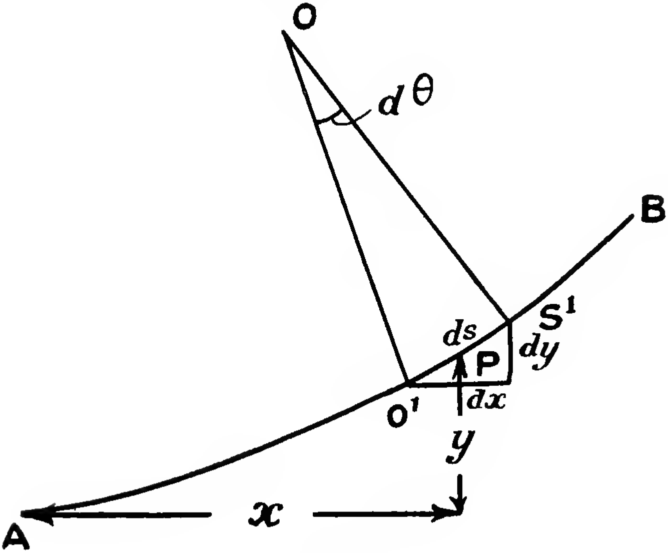

FIG. 91.

If $d\theta$ is the angle subtended at the centre by the element ds which is also an arc of the circle

$$ds = \mathrm{R}d\theta$$

$$\text{or } \mathrm{R} = \frac{ds}{d\theta} \quad . \quad . \quad . \quad (1)$$

The quantity $\frac{1}{\mathrm{R}}\left(= \frac{d\theta}{ds}\right)$ is called the *curvature* of the curve at the point P.

To get equation (1) in terms of ordinary or

Cartesian co-ordinates we put $\tan\theta = \frac{dy}{dx}$ (see § 4) and differentiating with regard to x

$$\frac{d(\tan\theta)}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$

$$\text{i.e. } \frac{d\tan\theta}{d\theta}\cdot\frac{d\theta}{ds} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\cdot\frac{dx}{ds}\,(*)$$

$$\text{i.e. } \sec^2\theta\,.\,\frac{d\theta}{ds} = \frac{d^2y}{dx^2}\,.\cos\theta$$

$$\text{i.e. } \frac{1}{\cos^2\theta}\cdot\frac{1}{\mathrm{R}} = \frac{d^2y}{dx^2}\,.\cos\theta$$

$$\text{i.e. } \frac{1}{\mathrm{R}} = \frac{d^2y}{dx^2}\cdot\cos^3\theta \qquad . \quad . \quad . \qquad (2)$$

$$\text{now } \cos\theta = \frac{dx}{ds} = \frac{dx}{\sqrt{dx^2+dy^2}} = \frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$

$$\therefore \cos^3\theta = \frac{1}{\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right)^3} = \frac{1}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

putting this in equation (2) we get

$$\frac{1}{\mathrm{R}} = \frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} \qquad . \quad . \quad . \qquad (3)$$

We stated on p. 210 that when a curve such as the form of a slightly deflected beam is very flat, its curvature is given approximately by $\frac{1}{\mathrm{R}} = \frac{d^2y}{dx^2}$; we can see from (3) that this is the case because for a very flat curve the slope $\frac{dy}{dx}$ will be very small, and if we neglect it altogether we get the result $\frac{1}{\mathrm{R}} = \frac{d^2y}{dx^2}$.

---

* On each side here we have an example of $\frac{dz}{dx} = \frac{dz}{dy}\cdot\frac{dy}{dx}$ [p 83].

Example of a Parabola.—Take the case of the parabola $y = 8x^2$ and find the radius of curvature at the origin and for the value of $x = 2$.

In this case $\frac{dy}{dx} = \frac{d8x^2}{dx} = 16x,$

$$\frac{d^2y}{dx^2} = 16,$$

$$\therefore \frac{1}{\mathrm{R}} = \frac{16}{\{1 + 256x^2\}^{\frac{3}{2}}},$$

$$\mathrm{R} = \frac{\{1 + 256x^2\}^{\frac{3}{2}}}{16}.$$

At the origin where $x = 0$ this gives $\mathrm{R} = \frac{1}{16}$; at the point where $x = 2$

$$\mathrm{R} = \frac{(1 + 1024)^{\frac{3}{2}}}{16}$$

$$= \frac{32^3}{16} \text{ nearly}$$

$$= 2048.$$

We shall give other applications later in considering some special curves.

§ 67. **The Length of Curved Outlines.**—Referring to Fig. 91, we see that if we assume that the length of a very short piece ds of the curve is equal to the length of the chord O′S′ we may write for the element of curve

$$ds^2 = dx^2 + dy^2$$

$$\text{i.e. } \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\therefore \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

$$\therefore ds = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx.$$

If therefore s is the length of the curve from the point A to the point B

$$s = \int_A^B \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or taking co-ordinates in the other direction

$$s = \int_A^B \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy \quad . \quad . \quad . \quad . \quad . \quad (2)$$

We shall have an example of this in dealing with the catenary.

§ 68. **The Catenary and the Tractrix.**—The catenary is the curve in which a uniform cable hangs when supporting its own weight only. Its equation is of the form

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)^{*} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where a is a constant and it may also be written

$$y = a \cosh \frac{x}{a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

---

[*] It is usual to write the expression $\frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ as $\cosh \frac{x}{a}$ and the expression $\frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right)$ as $\sinh \frac{x}{a}$, these being contracted forms for *hyperbolic cosine* and *hyperbolic sine* respectively, because there is an analogy between these functions with reference to a rectangular hyperbola and ordinary trigonmetrical functions with reference to a circle. It will be noted that $\cosh^2 \left(\frac{x}{a}\right) - \sinh^2 \left(\frac{x}{a}\right) = 1$. Values of these functions are tabulated on p. 239.

It follows that

$$\frac{d \cosh x}{dx} = \sinh x$$

$$\text{and } \frac{d \sinh x}{dx} = \cosh x.$$

This should be tested by the student.

The axis of x is called the *directrix* in this case. This curve has many interesting properties.

Take first the length of the curve from A to P (Fig. 92).

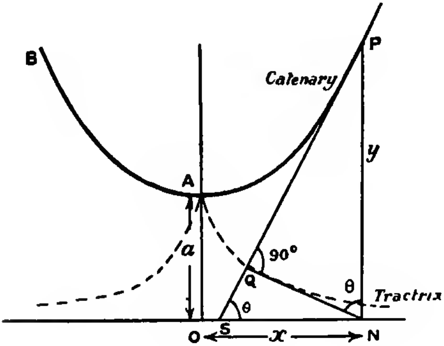

FIG. 92.

$$\frac{dy}{dx} = \frac{a}{2}\left(\frac{1}{a}\,e^{\frac{x}{a}} - \frac{1}{a}\,e^{-\frac{x}{a}}\right)$$

$$= \frac{1}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \quad . \quad . \quad . \quad (3)$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(e^{\frac{2x}{a}} + e^{-\frac{2x}{a}} - 2e^{\frac{x}{a}-\frac{x}{a}}\right)$$

$$= \frac{1}{4}\left(e^{\frac{2x}{a}} + e^{-\frac{2x}{a}}\right) - \frac{1}{2}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(e^{\frac{2x}{a}} + e^{-\frac{2x}{a}}\right) + \frac{1}{2}$$

$$= \left\{\frac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)\right\}^2$$

$$= \frac{y^2}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The length of the curve from A to P

$$= s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,.\,dx$$

$$= \int_0^x \frac{y}{a}\,dx$$

$$= \int_0^x \tfrac{1}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) dx$$

$$= \frac{a}{2}\left[e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right]_0^x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The expression vanishes at the lower limit, for $e^0 = 1$.

$$\therefore s = \frac{a}{2}\left(e^{\frac{x}{a}} - e^{-\frac{x}{a}}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

$$= a \sinh\left(\frac{x}{a}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Going back to equation (3) we see that

$$s = a \,.\, \frac{dy}{dx}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Referring to the figure, we note that

$$\frac{dy}{dx} = \tan\theta = \frac{\text{PN}}{\text{SN}}.$$

Now $1 + \tan^2\theta = \sec^2\theta$

$$\therefore \sec^2\theta = 1 + \left(\frac{dy}{dx}\right)^2 = \frac{y^2}{a^2} \text{ [from (4)]}$$

$$\therefore \sec\theta = \frac{y}{a}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now draw NQ perpendicular to the tangent PS.

Then $\text{NSP} = \theta = \frac{\pi}{2} - \text{SPN} = \text{PNQ}$,

and $\sec\theta = \frac{\text{PN}}{\text{QN}} = \frac{y}{\text{QN}} = \frac{y}{a}$ [from (9)].

$$\therefore \text{QN} = a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$\text{Also } \frac{\text{PQ}}{\text{QN}} = \tan\theta = \frac{dy}{dx}$$

$$\therefore \text{PQ} = a\frac{dy}{dx} = s \text{ [from (8)]}.$$

If, therefore, a string PQ held taut were wrapped round the catenary, Q would describe a curved path shown in dotted lines, and the point Q would come to the vertex A because PQ $= s$; this path is called the *tractrix*.

When two curves are so related that the second is the locus of the end of a string as it is unwound from the first, the second curve is called an *invòlute* of the first, so that the tractrix is an involute of the catenary.

We notice also that QN is tangential to the tractrix because it is at right angles to PQ and the instantaneous path of Q is perpendicular to PQ, and we have proved that its length is equal to a which is constant. The tractrix therefore has the property that its tangent is of constant length [see p. 179].

The tractrix gets its name from the fact that it is the curve moved through by a body placed at A and connected by a string of length a which is moved along ON so slowly that the body acquires no momentum — or the body may be supposed " rough ".

PROOF THAT A CABLE HANGS IN THE FORM OF A CATENARY.—Consider the equilibrium of a portion PA (Fig. 93) of the cable of length s; there are three forces acting on it, viz. a horizontal tension T_o at A, a tangential tension T at P, and a weight ws, s being the length of the cable from A to P, and w being the weight per unit length of the cable, the weight being assumed constant.

Then clearly from the triangle of forces $a\ b\ c$

$$ws = T_o \tan\theta = T_o \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{ws}{T_o}$$

$$\text{i.e. } s = \frac{T_o dy}{w\, dx}$$

$$= a \cdot \frac{dy}{dx} \quad \ldots \ldots \quad (1)$$

a being a constant because T^o and w are both constant.

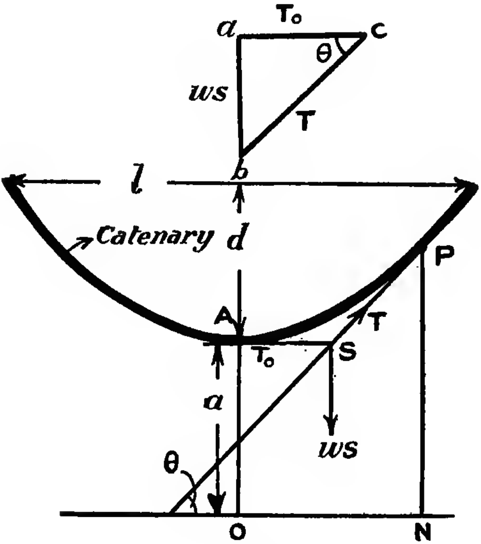

FIG. 93.

But we have already shown in equation (8) that this relation holds for the catenary, so that the catenary is a curve fulfilling the necessary conditions of stability of the cable.

We also see that since $a = \frac{T_o}{w}$, the horizontal tension T_o at A is w . AO.

Now consider the tension T at the point P.

It follows from the triangle of forces that

$$\frac{T}{T_o} = \sec\theta,$$

but from Fig. 92, $\sec\theta = \frac{PN}{QN} = \frac{y}{a}$

$$\therefore T = \frac{T_o y}{a}$$

$$= \frac{wa \,.\, y}{a}$$

$$= wy.$$

The tension therefore at any point of the cable is equal to the weight per unit length of the cable multiplied by the distance from the point to the directrix of the catenary.

Engineers usually regard the catenary as a troublesome curve to deal with in cable problems because if the span and dip are given the calculation of a, which is necessary before the curve can be drawn, is not obvious.

With the aid of tables of hyperbolic functions in the appendix, however, we can find a as follows:—

If l is the span of the cable and d is the dip,

$$y = a + d \text{ when } x = \frac{l}{2}$$

$$\therefore a + d = a \cosh\frac{l}{2a}$$

$$\therefore 1 + \frac{d}{a} = \cosh\frac{l}{2a}$$

$$\therefore \text{If } \frac{l}{2a} = z$$

$$1 + \frac{2d}{l} \,.\, z = \cosh z.$$

z can then be found as follows by plotting from the tables: a catenary $y = \cosh z$ (Fig. 94). Take for instance $\frac{l}{d} = 10$.

$$\text{Then } \frac{\cdot 2d}{l} = \frac{2}{10} = \cdot 2$$

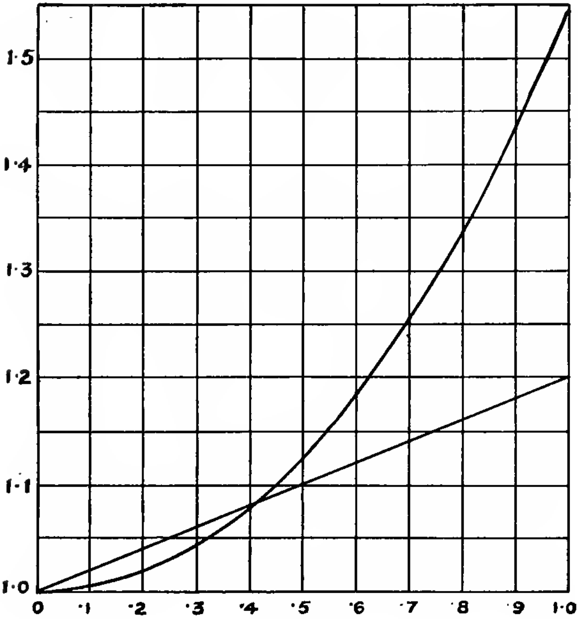

FIG. 94.

$\therefore$ draw the straight line $y = 1 + \cdot 2z$ starting from the point $y = 1$ as shown.

The straight line cuts the catenary at $z = \cdot 42$

$$\therefore \frac{l}{2a} \doteq \cdot 42 \text{ or } a = \frac{l}{\cdot 84} = 1{\cdot}19l.$$

If we are given the length $2s$ and span l and not the dip we use the relation

$$\begin{aligned} s &= a \sinh \frac{x}{a} \\ &= a \sinh \frac{l}{a} \end{aligned}$$

$$\text{i.e. } \frac{s}{l} = \frac{a}{l} \sinh \frac{l}{a}$$

$$\text{or if } \frac{s}{l} = b \text{ and } \frac{l}{a} = z$$

$$bz = \sinh z$$

b is given and so by plotting a curve of $\sinh z$ and drawing upon it the line representing bz we can find the solution in a similar manner.

§ 69. **Envelopes, Evolutes, and Involutes.**—The *envelope* of a series of curves of the same family, i.e. those obeying the same kind of law, may be defined as the locus of the points of intersection of the curves when such curves are infinitely close; it may be regarded as the boundary curve within or without which all the particular curves of the family lie.

A familiar example of this arises in the case of the geometrical construction shown in Fig. 95 for the parabola. The sides of the triangle are divided up into an equal number of parts and joined across as shown; the greater the number of parts, the nearer will the path of the successive intersection approach to the parabola which is thus the envelope of this particular family of straight lines.

There is also an example of envelopes in the case of the bending moment and shear diagrams for a rolling load upon a beam which will be found described in text-books upon the theory of structures.

The *evolute* of a curve is the locus or path of its centre of curvature. The circle is the curve whose centre of curvature is fixed so that the evolute of a circle is a fixed point, viz. the centre of the circle.

The centre of curvature may be regarded as the intersection of successive normals since the centre of curvature is the ultimate point of intersection of two close normals, the evolute is the envelope of the normals to the given curve. The normals to the original curve are therefore the tangents to the evolute.

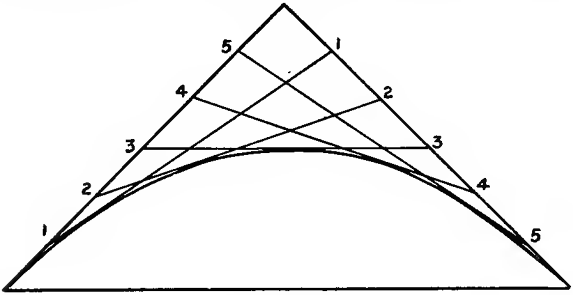

Fig. 95.

An *involute* of any given curve is the locus of points obtained by marking along the tangent to the curve at any point a length equal to the length of the curve from some fixed point upon it. We have already had an example of this in the case of the tractrix. If a curve A be the evolute of a curve B, B will be an involute of A.

A curve has an infinite number of involutes, one corresponding to each fixed point on the curve.

As a more practical definition of an involute, we may imagine a string to be wrapped round the given curve and a pencil fixed to a point along the string; if the string then be unwound while being kept taut, the pencil will draw an involute of the curve.

The original curve is the evolute of all its involutes.

THE CIRCULAR INVOLUTE.—The involute which has the greatest practical application is the circular involute which is used for the shape of gear wheel teeth. If we start unwinding at the point A (Fig. 96) we get the curve AQx, so that if SQ is the tangent to the circle at a point S, the length SQ will be equal to the arc AS.

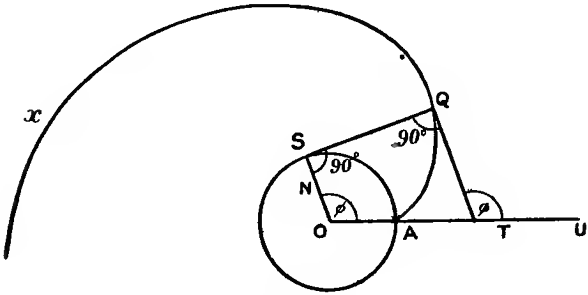

FIG. 96.

Since we take the starting-point A at any point on the curve there must be, as we have already stated, an infinite number of involutes to every curve, and in the case of the circle all the involutes will have the same shape.

At any point Q of the involute the centre of curvature is the corresponding point S on the circle because S is the instantaneous centre of movement of the string. We have already shown on p. 216, that

$$\frac{1}{R} = \frac{d\phi}{ds}.$$

The radius of curvature R in this case $=$ SQ $=$ length of arc AS $= r\phi$. Note also that the angle QTU through which the tangent to the involute has moved is also equal to ϕ

$$\therefore ds = Rd\phi = r\phi d\phi$$

When r is the radius of the circle

$$\therefore s = \text{length of arc AQ of the involute}$$

$$= \int_0^\phi r\phi d\phi = \frac{r\phi^2}{2}.$$

The constant of integration is zero because the curve starts from A where ϕ is zero.

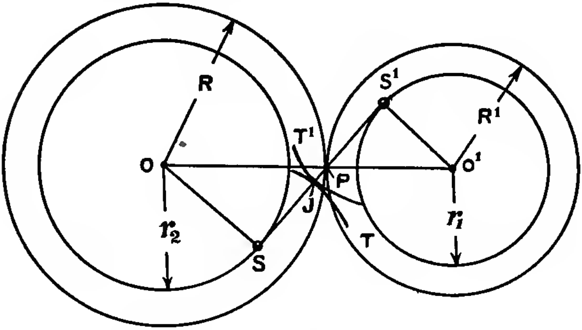

FIG. 97.

APPLICATION TO GEAR TEETH.—It is shown in books upon mechanism that for two tooth outlines to gear accurately their common normal must pass through a fixed point called the "pitch point". Let R, R_1 (Fig. 97) be the radii of the "pitch circles" of the wheels; let r_2, r_1 be the radii of circles called "base circles" which are so related that $\frac{r_2}{R} = \frac{r_1}{R_1}$, so that their common tangent SS′ passes through the pitch point P.

Let the outlines TT_1 be involutes of these base circles; then it is clear from the figure that SS′ is the common normal to the two curves at the point of contact J of the two teeth and SS′ passes through the pitch point P, so that the involute fulfils the necessary condition for gear teeth.

§ 70. **Cycloids.**—The cycloids are a series of curves which also arise in problems of toothed gearing.

The cycloid is the curve traced out by a point upon a circle which rolls without slip upon a straight line.

Referring to Fig. 98, if r is the radius of the rolling circle, the point P will describe the cycloid ACB, the base AB of which will be equal to $2\pi r$, the circumference of the circle.

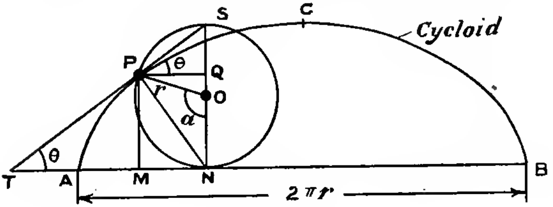

FIG. 98.

Take any position of the rolling circle.

Then the arc PN must be equal to the length AN if no slip occurs.

$$AN = r\alpha$$

$$MP = y = ON + OQ = r + r\cos(180 - \alpha)$$

$$y = r(1 - \cos\alpha) \quad . \quad . \quad . \quad . \quad (1)$$

$$\mathrm{MN} = \mathrm{PQ} = r \sin (180 - a) = r \sin a$$

$$\therefore \mathrm{AM} = x = \mathrm{AN} - \mathrm{MN}$$

$$x = r (a - \sin a) \quad . \quad . \quad . \quad . \quad (2)$$

Let PT be the tangent at the point P, then

$$\tan \theta = \frac{dy}{dx}$$

$$= \frac{dy}{da} \cdot \frac{da}{dx} = \frac{dy}{da} \div \frac{dx}{da}.$$

$$\text{Now } \frac{dy}{da} = \frac{d \{ r (1 - \cos a)}{da}$$

$$= r (0 + \sin a) = r \sin a \quad . \quad . \quad . \quad (3)$$

$$\frac{dx}{da} = \frac{d \{ r (a - \sin a)}{da}$$

$$= r (1 - \cos a) \quad . \quad . \quad . \quad . \quad (4)$$

$$\tan \theta = \frac{r \sin a}{r (1 - \cos a)}$$

$$= \frac{\sin a}{1 - \cos a} = \frac{2 \sin \frac{a}{2} \cos \frac{a}{2}}{2 \sin^2 \frac{a}{2}}$$

$$= \frac{\cos \frac{a}{2}}{\sin \frac{a}{2}} = \cot \frac{a}{2} \quad . \quad . \quad . \quad . \quad (5)$$

$$\text{but } \tan \theta = \cot (90 - \theta) = \cot \underline{\angle \mathrm{PSQ}}$$

$$\therefore \underline{\angle \mathrm{PSQ}} = \frac{a}{2}.$$

This means that S is on the diameter of the circle.

This can be seen more simply. Since the circle is rolling along AB, its point of contact N is the instantaneous centre about which the circle is rotating so that PN is the normal at P and PS the tangent to the curve, and since the angle in a semi-circle is a right angle NS must be a diameter of the circle.

Length of Arc of Cycloid.—We have shown on p. 219 that

$$ds^2 = dx^2 + dy^2$$

$$\therefore \left(\frac{ds}{da}\right)^2 = \left(\frac{dx}{da}\right)^2 + \left(\frac{dy}{da}\right)^2$$

$$= r^2 (1 - \cos a)^2 + r^2 \sin^2 a$$

$$= r^2 (1 - 2 \cos a + \cos^2 a + \sin^2 a)$$

$$= r^2 (2 - 2 \cos a) \text{ because } \sin^2 a + \cos^2 a = 1$$

$$= 2r^2 (1 - \cos a)$$

$$= 4r^2 \sin^2 \frac{a}{2}$$

$$\therefore \frac{ds}{da} = 2r \sin \frac{a}{2} \quad . \quad . \quad . \quad . \quad (6)$$

$$\therefore \text{length of arc AP} = s = \int_0^a 2r \sin \frac{a}{2}\, da$$

$$= -2r \left[2 \cos \frac{a}{2}\right]^a$$

$$= -2r \left(2 \cos \frac{a}{2} - 2\right)$$

$$= 4r \left(1 - \cos \frac{a}{2}\right) \quad . \quad . \quad . \quad (7)$$

$\therefore$ Length of whole curve AB for which $a = 360°$

$$= 4r \{1 - (-1)\} = 8r \quad . \quad . \quad . \quad (8)$$

Radius of Curvature of Cycloid.—

$$R = \frac{ds}{d\theta}$$

$$= \frac{ds}{da} \cdot \frac{da}{d\theta} = 2r \sin \frac{a}{2} \div \frac{d\theta}{da} \text{ [from (6)].}$$

Now we have shown that $\theta = 90 - \dfrac{a}{2}$

$$\text{and } \frac{d\theta}{da} = -\tfrac{1}{2}$$

$$\begin{aligned} \therefore R &= -2r\cos\theta \div \tfrac{1}{2} \\ &= -4r\cos\theta \\ &= -2\cdot 2r\cos \mathrm{PNS} \\ &= -2\mathrm{PN} \quad . \quad . \quad . \qquad (9) \end{aligned}$$

The sign may be adjusted by convention; note that either sign may be attached to formula (6).

THE TROCHOID.—The path generated by some point fixed relatively to the circle but not situated on the circumference is called a trochoid. If the point is outside the circle, the trochoid has loops, as shown in Fig. 99.

EPICYCLOID AND HYPOCYCLOID.—When a circle rolls outside another circle without slipping, the resulting curve is called an epicycloid, while if it rolls inside, it is called a hypocycloid. The curves are shown on Fig. 100.

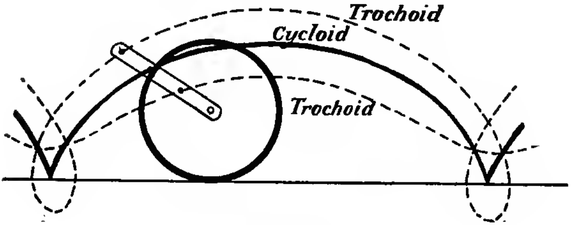

FIG. 99.

If the radius R of the main circle is an exact multiple (2, 3, 4, etc.) of the radius r of the rolling circle, the curves will repeat as the rolling circle continues to roll more than once round the main circle. If the rolling circle has a radius equal to half that of the main circle, the hypocycloid becomes the diameter. This can be proved from Fig. 101 in which A represents the starting-point and OP represents any given position of the diameter of the rolling circle.

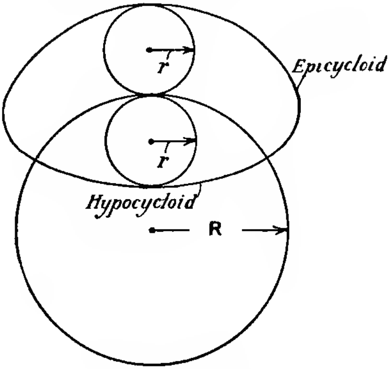

FIG. 100.

Then if $\angle AOP = \theta$, $\angle QSP = 2\theta$ because the

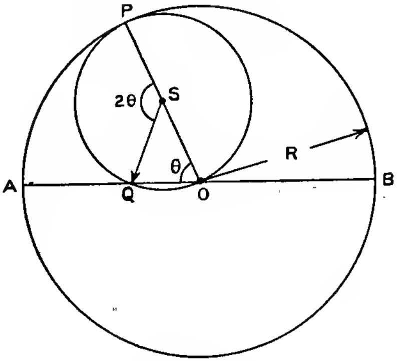

FIG. 101.

angle at the centre of a circle is twice that at the circumference.

Now arc AP $= R\theta$
and arc QP $= r \cdot 2\theta$, so since $R = 2r$
arc AP $=$ arc PQ

$\therefore$ Q which is on the diameter AB is also on the hypocycloid.

APPLICATION TO TOOTHED GEARING.—Referring to Fig. 102, let the pitch circles APB and CPD touch at P and let *ab* and *cd* be portions of hypocycloidal and epicycloidal tooth outlines touching at Q, the common rolling circle being QPR.

ab is the curve for rolling upon APB and *dc* is that for rolling upon CPD. At the instant, the circle QPR is rolling outside CPD and inside APB and the direction of each curve must be normal to the line QP because this circle is rolling about P as an instantaneous centre.

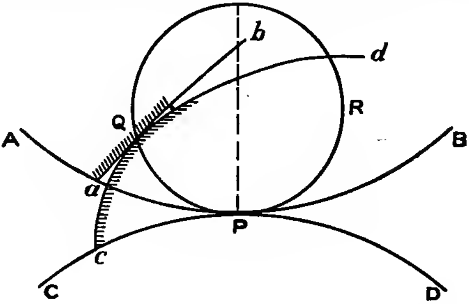

FIG. 102.

The two curves therefore fulfil the necessary condition that the common normal shall pass through the pitch point P.

These outlines determine only the root of one tooth and the point of the other; the other portions of each teeth can be obtained similarly but need not necessarily have the same radius of rolling circle as the first portions.

Cycloidal teeth are now seldom used in practice.

EXERCISES 10.

1. Find the radius of curvature at the point $x=1$ of the curve $y^2 = x^3$.

2. Find the radius of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$.

3. Find the radius of curvature at the point $x=2$ of the hyperbola $xy = 4$.

4. Find the radius of curvature of the catenary $y=a \cosh \frac{x}{a}$.

5. Find the radius of curvature of the cubic parabola $ay^2 = x^3$.

6. Find an expression for the length of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

7. Find the length of the curve $9\,ay^2 = 4x^3$ starting from the point $x = o$.

8. Draw a curve $y = \sinh x$ and find the value of a for a cable whose length is twice the span. (See p. 226.)

9. Find the length of the parabola $y^2 = 4\,ax$ from the vertex.

10. If R is the radius of the fixed circle and r that of the rolling circle, show that the co-ordinates of a point on the epicycloid may be written :—

$$x = (R + r) \cos \alpha + b \cos \frac{(R + r)}{r} \alpha$$

$$y = (R + r) \sin \alpha + b \sin \frac{(R + r)}{r} \alpha$$

α being the angle turned through by the point of contact.

ANSWERS TO EXERCISES.

EXERCISES 10.

1. $\frac{13^{\frac{3}{2}}}{6} = 7{\cdot}81.$ 2. $-b^4x^2.$ 3. $2\sqrt{2}.$ 4. $\frac{y^2}{a}.$

5. $\frac{a^2}{6x}\left(1+\frac{9x^4}{a^4}\right)^{\frac{3}{2}}.$ 6. $\frac{3a^{\frac{1}{3}}\,x^{\frac{2}{3}}}{2}.$

7. $\frac{2a}{3}\left\{\left(1+\frac{x}{a}\right)^{\frac{3}{2}}-1\right\}.$ 8. $a = {\cdot}455l.$

9. $s = \sqrt{ax+x^2} + a\log\left(\frac{\sqrt{x}+\sqrt{a+x}}{\sqrt{a}}\right).$

EXPONENTIAL AND HYPERBOLIC FUNCTIONS.

x	e^x	e^{-x}	cosh x	sinh x
0	1·000	1·000	1·000	0
·1	1·105	·905	1·005	·100
·2	1·221	·819	1·020	·201
·3	1·350	·741	1·045	·305
·4	1·492	·670	1·081	·411
·5	1·649	·607	1·128	·521
·6	1·822	·549	1·185	·637
·7	2·014	·497	1·255	·759
·8	2·226	·449	1·337	·888
·9	2·460	·407	1·433	1·027
1·0	2·718	·368	1·543	1·175
1·1	3·004	333	1·669	1·336
1·2	3·320	·301	1·811	1·509
1·3	3·669	·273	1·971	1·698
1·4	4·055	·247	2·151	1·904
1·5	4·482	·223	2·352	2·129
1·6	4·953	·202	2·577	2·376
1·7	5·474	·183	2·828	2·646
1·8	6·050	·165	3·107	2·942
1·9	6·686	·150	3·418	3·268
2·0	7·389	·135	3·762	3·627
2·1	8·166	·122	4·144	4·022
2·2	9·025	·111	4·568	4·457
2·3	9·974	·100	5·037	4·937
2·4	11·023	·091	5·557	5·466
2·5	12·182	·082	6·132	6·050

HYPERBOLIC LOGARITHMS OF NUMBERS FROM 1 TO 50.

No.	Log.	No.	Log.	No.	Log.	No.	Log.
1·00	·0000	1·40	·3365	1·80	·5878	2·20	·7885
1·01	·0099	1·41	·3436	1·81	·5933	2·21	·7930
1·02	·0198	1·42	·3507	1·82	·5988	2·22	·7975
1·03	·0296	1·43	·3577	1·83	·6043	2·23	·8020
1·04	·0392	1·44	·3646	1·84	·6098	2·24	·8065
1·05	·0488	1·45	·3716	1·85	·6152	2·25	·8109
1·06	·0583	1·46	·3784	1·86	·6206	2·26	·8154
1·07	·0677	1·47	·3853	1·87	·6259	2·27	·8198
1·08	·0770	1·48	·3920	1·88	·6313	2·28	·8242
1·09	·0862	1·49	·3988	1·89	·6366	2·29	·8286
1·10	·0953	1·50	·4055	1·90	·6419	2·30	·8329
1·11	·1044	1·51	·4121	1·91	·6471	2·31	·8372
1·12	·1133	1·52	·4187	1·92	·6523	2·32	·8416
1·13	·1222	1·53	·4253	1·93	·6575	2·33	·8458
1·14	·1310	1·54	·4318	1·94	·6627	2·34	·8502
1·15	·1398	1·55	·4383	1·95	·6678	2·35	·8544
1·16	·1484	1·56	·4447	1·96	·6729	2·36	·8587
1·17	·1570	1·57	·4511	1·97	·6780	2·37	·8629
1·18	·1655	1·58	·4574	1·98	·6831	2·38	·8671
1·19	·1740	1·59	·4637	1·99	·6881	2·39	·8713
1·20	·1823	1·60	·4700	2·00	·6931	2·40	·8755
1·21	·1906	1·61	·4762	2·01	·6981	2·41	·8796
1·22	·1988	1·62	·4824	2·02	·7031	2·42	·8838
1·23	·2070	1·63	·4886	2·03	·7080	2·43	·8879
1·24	·2151	1·64	·4947	2·04	·7129	2·44	·8920
1·25	·2231	1·65	·5008	2·05	·7178	2·45	·8961
1·26	·2311	1·66	·5068	2·06	·7227	2·46	·9002
1·27	·2390	1·67	·5128	2·07	·7275	2·47	·9042
1·28	·2469	1·68	·5188	2·08	·7324	2·48	·9083
1·29	·2546	1·69	·5247	2·09	·7372	2·49	·9123
1·30	·2624	1·70	·5306	2·10	·7419	2·50	·9163
1·31	·2700	1·71	·5365	2·11	·7467	2·51	·9203
1·32	·2776	1·72	·5423	2·12	·7514	2·52	·9243
1·33	·2852	1·73	·5481	2·13	·7561	2·53	·9282
1·34	·2927	1·74	·5539	2·14	·7608	2·54	·9322
1·35	·3001	1·75	·5596	2·15	·7655	2·55	·9361
1·36	·3075	1·76	·5653	2·16	·7701	2·56	·9400
1·37	·3148	1·77	·5710	2·17	·7747	2·57	·9439
1·38	·3221	1·78	·5766	2·18	·7793	2·58	·9478
1·39	·3293	1·79	·5822	2·19	·7839	2·59	·9517

No.	Log.	No.	Log.	No.	Log.	No.	Log.
2·60	·9555	3·04	1·1119	3·48	1·2470	3·92	1·3661
2·61	·9594	3·05	1·1151	3·49	1·2499	3·93	1·3686
2·62	·9632	3·06	1·1184	3·50	1·2528	3·94	1·3712
2·63	·9670	3·07	1·1217	3·51	1·2556	3·95	1·3737
2·64	·9708	3·08	1·1249	3·52	1·2585	3·96	1·3762
2·65	·9746	3·09	1·1282	3·53	1·2613	3·97	1·3788
2·66	·9783	3·10	1·1314	3·54	1·2641	3·98	1·3813
2·67	·9821	3·11	1·1346	3·55	1·2669	3·99	1·3838
2·68	·9858	3·12	1·1378	3·56	1·2698	4·00	1·3863
2·69	·9895	3·13	1·1410	3·57	1·2726	4·01	1·3888
2·70	·9933	3·14	1·1442	3·58	1·2754	4·02	1·3913
2·71	·9969	3·15	1·1474	3·59	1·2782	4·03	1·3938
2·72	1·0006	3·16	1·1506	3·60	1·2809	4·04	1·3962
2·73	1·0043	3·17	1·1537	3·61	1·2837	4·05	1·3987
2·74	1·0080	3·18	1·1569	3·62	1·2865	4·06	1·4012
2·75	1·0116	3·19	1·1600	3·63	1·2892	4·07	1·4036
2·76	1·0152	3·20	1·1632	3·64	1·2920	4·08	1·4061
2·77	1·0188	3·21	1·1663	3·65	1·2947	4·09	1·4085
2·78	1·0225	3·22	1·1694	3·66	1·2975	4·10	1·4110
2·79	1·0260	3·23	1·1725	3·67	1·3002	4·11	1·4134
2·80	1·0296	3·24	1·1756	3·68	1·3029	4·12	1·4159
2·81	1·0332	3·25	1·1787	3·69	1·3056	4·13	1·4183
2·82	1·0367	3·26	1·1817	3·70	1·3083	4·14	1·4207
2·83	1·0403	3·27	1·1848	3·71	1·3110	4·15	1·4231
2·84	1·0438	3·28	1·1878	3·72	1·3137	4·16	1·4255
2·85	1·0473	3·29	1·1909	3·73	1·3164	4·17	1·4279
2·86	1·0508	3·30	1·1939	3·74	1·3191	4·18	1·4303
2·87	1·0543	3·31	1·1969	3·75	1·3218	4·19	1·4327
2·88	1·0578	3·32	1·1999	3·76	1·3244	4·20	1·4351
2·89	1·0613	3·33	1·2030	3·77	1·3271	4·21	1·4375
2·90	1·0647	3·34	1·2060	3·78	1·3297	4·22	1·4398
2·91	1·0682	3·35	1·2090	3·79	1·3324	4·23	1·4422
2·92	1·0716	3·36	1·2119	3·80	1·3350	4·24	1·4446
2·93	1·0750	3·37	1·2149	3·81	1·3376	4·25	1·4469
2·94	1·0784	3·38	1·2179	3·82	1·3403	4·26	1·4493
2·95	1·0813	3·39	1·2208	3·83	1·3429	4·27	1·4516
2·96	1·0852	3·40	1·2238	3·84	1·3455	4·28	1·4540
2·97	1·0886	3·41	1·2267	3·85	1·3481	4·29	1·4563
2·98	1·0919	3·42	1·2296	3·86	1·3507	4·30	1·4586
2·99	1·0953	3·43	1·2326	3·87	1·3533	4·31	1·4609
3·00	1·0986	3·44	1·2355	3·88	1·3558	4·32	1·4633
3·01	1·1019	3·45	1·2384	3·89	1·3584	4·33	1·4656
3·02	1·1053	3·46	1·2413	3·90	1·3610	4·34	1·4679
3·03	1·1086	3·47	1·2442	3·91	1·3635	4·35	1·4702

No.	Log.	No.	Log.	No.	Log.	No.	Log.
4·36	1·4725	4·80	1·5686	5·24	1·6563	5·68	1·7370
4·37	1·4748	4·81	1·5707	5·25	1·6582	5·69	1·7387
4·38	1·4770	4·82	1·5728	5·26	1·6601	5·70	1·7405
4·39	1·4793	4·83	1·5748	5·27	1·6620	5·71	1·7422
4·40	1·4816	4·84	1·5769	5·28	1·6639	5·72	1·7440
4·41	1·4839	4·85	1·5790	5·29	1·6658	5·73	1·7457
4·42	1·4861	4·86	1·5810	5·30	1·6677	5·74	1·7475
4·43	1·4884	4·87	1·5831	5·31	1·6696	5·75	1·7492
4·44	1·4907	4·88	1·5851	5·32	1·6715	5·76	1·7509
4·45	1·4929	4·89	1·5872	5·33	1·6734	5·77	1·7527
4·46	1·4951	4·90	1·5892	5·34	1·6752	5·78	1·7544
4·47	1·4974	4·91	1·5913	5·35	1·6771	5·79	1·7561
4·48	1·4996	4·92	1·5933	5·36	1·6790	5·80	1·7579
4·49	1·5019	4·93	1·5953	5·37	1·6808	5·81	1·7596
4·50	1·5041	4·94	1·5974	5·38	1·6827	5·82	1·7613
4·51	1·5063	4·95	1·5994	5·39	1·6845	5·83	1·7630
4·52	1·5085	4·96	1·6014	5·40	1·6864	5·84	1·7647
4·53	1·5107	4·97	1·6034	5·41	1·6882	5·85	1·7664
4·54	1·5129	4·98	1·6054	5·42	1·6901	5·86	1·7681
4·55	1·5151	4·99	1·6074	5·43	1·6919	5·87	1·7699
4·56	1·5173	5·00	1·6094	5·44	1·6938	5·88	1·7716
4·57	1·5195	5·01	1·6114	5·45	1·6956	5·89	1·7733
4·58	1·5217	5·02	1·6134	5·46	1·6974	5·90	1·7750
4·59	1·5239	5·03	1·6154	5·47	1·6993	5·91	1·7766
4·60	1·5261	5·04	1·6174	5·48	1·7011	5·92	1·7783
4·61	1·5282	5·05	1·6194	5·49	1·7029	5·93	1·7800
4·62	1·5304	5·06	1·6214	5·50	1·7047	5·94	1·7817
4·63	1·5326	5·07	1·6233	5·51	1·7066	5·95	1·7834
4·64	1·5347	5·08	1·6253	5·52	1·7084	5·96	1·7851
4·65	1·5369	5·09	1·6273	5·53	1·7102	5·97	1·7867
4·66	1·5390	5·10	1·6292	5·54	1·7120	5·98	1·7884
4·67	1·5412	5·11	1·6312	5·55	1·7138	5·99	1·7901
4·68	1·5433	5·12	1·6332	5·56	1·7156	6·00	1·7918
4·69	1·5454	5·13	1·6351	5·57	1·7174	6·01	1·7934
4·70	1·5476	5·14	1·6371	5·58	1·7192	6·02	1·7951
4·71	1·5497	5·15	1·6390	5·59	1·7210	6·03	1·7967
4·72	1·5518	5·16	1·6409	5·60	1·7228	6·04	1·7984
4·73	1·5539	5·17	1·6429	5·61	1·7246	6·05	1·8001
4·74	1·5560	5·18	1·6448	5·62	1·7263	6·06	1·8017
4·75	1·5581	5·19	1·6467	5·63	1·7281	6·07	1·8034
4·76	1·5602	5·20	1·6487	5·64	1·7299	6·08	1·8050
4·77	1·5623	5·21	1·6506	5·65	1·7317	6·09	1·8066
4·78	1·5644	5·22	1·6525	5·66	1·7334	6·10	1·8083
4·79	1·5665	5·23	1·6544	5·67	1·7352	6·11	1·8099

No.	Log.	No.	Log.	No.	Log.	No.	Log.
6·12	1·8116	6 56	1·8810	7·00	1·9459	7·44	2·0069
6·13	1·8132	6·57	1·8825	7·01	1·9473	7·45	2·0082
6·14	1·8148	6·58	1·8840	7·02	1·9488	7·46	2·0096
6·15	1·8165	6·59	1·8856	7·03	1·9502	7·47	2·0109
6·16	1·8181	6·60	1·8871	7·04	1·9516	7·48	2·0122
6·17	1·8197	6·61	1·8886	7·05	1·9530	7·49	2·0136
6·18	1·8213	6·62	1·8901	7·06	1·9544	7·50	2·0149
6·19	1·8229	6·63	1·8916	7·07	1·9559	7·51	2·0162
6·20	1·8245	6·64	1 8931	7·08	1·9573	7·52	2·0176
6·21	1·8262	6·65	1·8946	7·09	1·9587	7·53	2·0189
6·22	1·8278	6·66	1·8961	7·10	1·9601	7·54	2·0202
6·23	1·8294	6·67	1·8976	7·11	1·9615	7·55	2·0215
6·24	1·8310	6·68	1·8991	7·12	1·9629	7·56	2·0229
6·25	1·8326	6·69	1·9006	7·13	1·9643	7·57	2·0242
6·26	1 8342	6·70	1·9021	7·14	1 9657	7·58	2·0255
6·27	1·8358	6·71	1·9036	7·15	1·9671	7·59	2·0268
6·28	1·8374	6·72	1·9051	7·16	1·9685	7·60	2·0281
6·29	1·8390	6·73	1·9066	7·17	1·9699	7·61	2·0295
6·30	1·8405	6·74	1·9081	7·18	1·9713	7·62	2·0308
6·31	1·8421	6·75	1·9095	7·19	1·9727	7·63	2·0321
6·32	1·8437	6·76	1·9110	7·20	1·9741	7·64	2·0334
6·33	1·8453	6·77	1·9125	7·21	1·9755	7·65	2·0347
6·34	1·8469	6·78	1·9140	7·22	1·9769	7·66	2·0360
6·35	1·8485	6·79	1·9155	7·23	1·9782	7·67	2·0373
6·36	1·8500	6·80	1·9169	7·24	1·9796	7·68	2·0386
6·37	1·8516	6·81	1·9184	7·25	1·9810	7·69	2·0399
6·38	1·8532	6·82	1·9199	7·26	1·9824	7·70	2·0412
6·39	1·8547	6·83	1·9213	7·27	1·9838	7·71	2·0425
6·40	1·8563	6·84	1·9228	7·28	1·9851	7·72	2·0438
6·41	1·8579	6·85	1·9242	7·29	1·9865	7·73	2·0451
6·42	1·8594	6·86	1·9257	7·30	1·9879	7·74	2·0464
6·43	1·8610	6·87	1·9272	7·31	1·9892	7·75	2·0477
6·44	1·8625	6·88	1·9286	7·32	1·9906	7·76	2·0490
6·45	1·8641	6·89	1·6301	7·33	1·9920	7·77	2·0503
6·46	1·8656	6·90	1·9315	7·34	1·9933	7·78	2·0516
6·47	1·8672	6·91	1·9330	7·35	1·9947	7·79	2·0528
6·48	1·8687	6·92	1·9344	7·36	1·9961	7·80	2·0541
6·49	1·8703	6·93	1·9359	7·37	1·9974	7·81	2·0554
6·50	1·8718	6·94	1·9373	7·38	1·9988	7·82	2·0567
6·51	1·8733	6·95	1·9387	7·39	2·0001	7·83	2·0580
6·52	1·8749	6·96	1·9402	7·40	2·0015	7·84	2·0592
6·53	1·8764	6·97	1·9416	7·41	2·0028	7·85	2·0605
6·54	1·8779	6·98	1·9430	7·42	2·0042	7·86	2·0618
6·55	1·8795	6·99	1·9445	7·43	2·0055	7·87	2·0631

No.	Log.	No.	Log.	No.	Log.	No.	Log.
7·88	2·0643	8·32	2·1187	8·76	2·1702	9·20	2·2192
7·89	2·0656	8·33	2·1199	8·77	2·1713	9·21	2·2203
7·90	2·0669	8·34	2·1211	8·78	2·1725	9·22	2·2214
7·91	2·0681	8·35	2·1223	8·79	2·1736	9·23	2·2225
7·92	2·0694	8·36	2·1235	8·80	2·1748	9·24	2·2235
7·93	2·0707	8·37	2·1247	8·81	2·1759	9·25	2·3246
7·94	2·0719	8·38	2·1258	8·82	2·1770	9·26	2·2257
7·95	2·0732	8·39	2·1270	8·83	2·1782	9·27	2·2268
7·96	2·0744	8·40	2·1282	8·84	2·1793	9·28	2·2279
7·97	2·0757	8·41	2·1294	8·85	2·1804	9·29	2·2289
7·98	2·0769	8·42	2·1306	8·86	2·1815	9·30	2·2300
7·99	2·0782	8·43	2·1318	8·87	2·1827	9·31	2·2311
8·00	2·0794	8·44	2·1330	8·88	2·1838	9·32	2·2322
8·01	2·0807	8·45	2·1342	8·89	2·1849	9·33	2·2332
8·02	2·0819	8·46	2·1353	8·90	2·1861	9·34	2·2343
8·03	2·0832	8·47	2·1365	8·91	2·1872	9·35	2·2354
8·04	2·0844	8·48	2·1377	8·92	2·1883	9·36	2·2364
8·05	2·0857	8·49	2·1389	8·93	2·1894	9·37	2·2375
8·06	2·0869	8·50	2·1401	8·94	2·1905	9·38	2·2386
8·07	2·0882	8·51	2·1412	8·95	2·1917	9·39	2·2396
8·08	2·0894	8·52	2·1424	8·96	2·1928	9·40	2·2407
8·09	2·0906	8·53	2·1436	8·97	2·1939	9·41	2·2418
8·10	2·0919	8·54	2·1448	8·98	2·1950	9·42	2·2428
8·11	2·0931	8·55	2·1459	8·99	2·1961	9·43	2·2439
8·12	2·0943	8·56	2·1471	9·00	2·1972	9·44	2·2450
8·13	2·0956	8·57	2·1483	9·01	2·1983	9·45	2·2460
8·14	2·0968	8·58	2·1494	9·02	2·1994	9·46	2·2471
8·15	2·0980	8·59	2·1506	9·03	2·2006	9·47	2·2481
8·16	2·0992	8·60	2·1518	9·04	2·2017	9·48	2·2492
8·17	2·1005	8·61	2·1529	9·05	3·2028	9·49	2·2502
8·18	2·1017	8·62	2·1541	9·06	2·2039	9·50	2·2513
8·19	2·1029	8·63	2·1552	9·07	2·2050	9·51	2·2523
8·20	2·1041	8·64	2·1564	9·08	2·2061	9·52	2·2534
8·21	2·1054	8·65	2·1576	9·09	2·2072	9·53	2·2544
8·22	2·1066	8·66	2·1587	9·10	2·2083	9·54	2·2555
8·23	2·1078	8·67	2·1599	9·11	2·2094	9·55	2·2565
8·24	2·1090	8·68	2·1610	9·12	2·2105	9·56	2·2576
8·25	2·1102	8·69	2·1622	9·13	2·2116	9·57	2·2586
8·26	2·1114	8·70	2·1633	9·14	2·2127	9·58	2·2597
8·27	2·1126	8·71	2·1645	9·15	2·2138	9·59	2·2607
8·28	2·1138	8·72	2·1656	9·16	2·2148	9·60	2·2618
8·29	2·1150	8·73	2·1668	9·17	2·2159	9·61	2·2628
8·30	2·1163	8·74	2·1679	9·18	2·2170	9·62	2·2638
8·31	2·1175	8·75	2·1691	9·19	2·2181	9·63	2·2649

No.	Log.	No.	Log.	No.	Log.	No.	Log.
9·64	2·2659	9·89	2·2915	13·50	2·6027	29·00	3·3673
9·65	2·2670	9·90	2·2925	13·75	2·6211	30·00	3·4012
9·66	2·2680	9·91	2·2935	14·00	2·6391	31·00	3·4340
9·67	2·2690	9·92	2·2946	14·25	2·6567	32·00	3·4657
9·68	2·2701	9·93	2·2956	14·50	2·6740	33·00	3·4965
9·69	2·2711	9·94	2·2966	14·75	2·6913	34·00	3·5263
9·70	2·2721	9·95	2·2976	15·00	2·7081	35·00	3·5553
9·71	2·2732	9·96	2·2986	15·50	2·7408	36·00	3·5835
9·72	2·2742	9·97	2·2996	16·00	2·7726	37·00	3·6109
9·73	2·2752	9·98	2·3006	16·50	2·8034	38·00	3·6376
9·74	2·2762	9·99	2·3016	17·00	2·8332	39·00	3·6636
9·75	2·2773	10·00	2·3026	17·50	2·8621	40·00	3·6889
9·76	2·2783	10·25	2·3279	18·00	2·8904	41·00	3·7136
9·77	2·2793	10·50	2·3513	18·50	2·9173	42·00	3·7377
9·78	2·2803	10·75	2·3749	19·00	2·9444	43·00	3·7612
9·79	2·2814	11·00	2·3979	19·50	2·9703	44·00	3·7842
9·80	2·2824	11·25	2·4201	20·00	2·9957	45·00	3·8067
9·81	2·2834	11·50	2·4430	21·00	3·0445	46·00	3·8286
9·82	2·2844	11·75	2·4636	22·00	3·0911	47·00	3·8501
9·83	2·2854	12·00	2·4849	23·00	3·1355	48·00	3·8712
9·84	2·2865	12·25	2·5052	24·00	3·1781	49·00	3·8918
9·85	2·2875	12·50	2·5262	25·00	3·2189	50·00	3·9120
9·86	2·2885	12·75	2·5455	26·00	3·2581		
9·87	2·2895	13·00	2·5649	27·00	3·2958		
9·88	2·2905	13·25	2·5840	28·00	3·3322		

TABLES.

LOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170						4	9	13	17	21	26	30	34	38
						0212	0253	0294	0334	0374	4	8	12	16	20	24	28	32	37
11	0414	0453	0492	0531	0569						4	8	12	15	19	23	27	31	35
						0607	0645	0682	0719	0755	4	7	11	15	19	22	26	30	33
12	0792	0828	0864	0899	0934	0969					3	7	11	14	18	21	25	28	32
							1004	1038	1072	1106	3	7	10	14	17	20	24	27	31
13	1139	1173	1206	1239	1271						3	7	10	13	16	20	23	26	30
						1303	1335	1367	1399	1430	3	7	10	12	16	19	22	25	29
14	1461	1492	1523	1553							3	6	9	12	15	18	21	24	28
					1584	1614	1644	1673	1703	1732	3	6	9	12	15	17	20	23	26
15	1761	1790	1818	1847	1875	1903					3	6	9	11	14	17	20	23	26
							1931	1959	1987	2014	3	5	8	11	14	16	19	22	25
16	2041	2068	2095	2122	2148						3	5	8	11	14	16	19	22	24
						2175	2201	2227	2253	2279	3	5	8	10	13	15	18	21	23
17	2304	2330	2355	2380	2405	2430					3	5	8	10	13	15	18	20	23
							2455	2480	2504	2529	2	5	7	10	12	15	17	19	22
18	2553	2577	2601	2625	2648						2	5	7	9	12	14	16	19	21
						2672	2695	2718	2742	2765	2	5	7	9	11	14	16	18	21
19	2788	2810	2833	2856	2878						2	4	7	9	11	13	16	18	20
						2900	2923	2945	2967	2989	2	4	6	8	11	13	15	17	19
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16

25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8

LOGARITHMS.—*Continued.*

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5

75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	2	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4

ANTILOGARITHMS.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1067	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	2	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	2	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	2	2	2	3	3
14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	2	2	2	3	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	2	2	2	3	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	2	2	2	3	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	2	2	2	3	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	2	2	2	3	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	2	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4

·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	2	2	3	3	4	4
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	2	2	3	3	4	4
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	2	2	3	3	4	4
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	2	2	3	3	4	4
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	2	2	3	3	4	4
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	2	2	3	3	4	4
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	3	3	4	4	5
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	3	3	4	4	5
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	3	3	4	4	5
·37	3344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	3	3	4	4	5
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2459	1	1	2	2	3	3	4	4	5
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	3	3	4	5	5
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	3	4	4	5	5
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	3	3	4	5	5	6
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6

ANTILOGARITHMS.—*Continued.*

	0	1	2	3	4	5	0	7	8	9	1	2	3	4	5	6	7	8	9
·50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
·51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
·52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
·53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
·54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
·55	3548	3556	3565	3573	3581	3580	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
·56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
·57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
·58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
·59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
·60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
·61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
·62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
·63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
·64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
·65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
·66	4571	4581	4692	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
·67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
·68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
·69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
·70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
·71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
·72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
·73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
·74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12

·75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
·76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
·77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
·78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
·79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
·80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
·81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
·82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
·83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
·84	6918	6934	6950	6966	6982	6998	7515	7031	7047	7063	2	3	5	6	8	10	11	13	15
·85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
·86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
·87	7413	7430	7447	7464	7482	7499	2516	7534	7551	7568	2	3	5	7	9	10	12	14	16
·88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
·89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
·90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
·91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
·92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
·93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
·94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
·95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
·96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
·97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
·98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
·99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

TRIGONOMETRICAL FUNCTIONS.

Angle.		Chord.	Sine.	Tangent.	Co-tangent.	Cosine.			
Degrees.	Radians.								
0°	0	0	0	0	∞	1	1·414	1·5708	90°
1	·0175	·017	·0175	·0175	57·2900	·9998	1·402	1·5533	89
2	·0349	·035	·0349	·0349	28·6363	·9994	1·389	1·5359	88
3	·0524	·052	·0523	·0524	19·0811	·9986	1·377	1·5184	87
4	·0698	·070	·0698	·0699	14·3007	·9976	1·364	1·5010	86
5	·0873	·087	0·872	·0875	11·4301	·9962	1·351	1·4835	85
6	·1047	·105	·1045	·1051	9·5144	·9945	1·338	1·4661	84
7	·1222	·122	·1219	·1228	8·1443	·9925	1·325	1·4486	83
8	·1396	·140	·1392	·1405	7·1154	·9903	1·312	1·4312	82
9	·1571	·157	·1564	·1584	6·3138	·9877	1·299	1·4137	81
10	·1745	·174	·1736	·1763	5·6713	·9848	1·286	1·3963	80
11	·1920	·192	·1908	·1944	5·1446	·9816	1·272	1·3788	79
12	·2094	·209	·2079	·2126	4·7046	·9781	1·259	1·3614	78
13	·2269	·226	·2250	·2309	4·3315	·9744	1·245	1·3439	77
14	·2443	·244	·2419	·2493	4·0108	·9703	1·231	1·3265	76
15	·2618	·261	·2588	·2679	3·7321	·9659	1·218	1·3090	75
16	·2793	·278	·2756	·2867	3·4874	·9613	1·204	1·2915	74
17	·2967	·296	·2924	·3057	3·2709	·9563	1·190	1·2741	73
18	·3142	·313	·3090	·3249	3·0777	·9511	1·176	1·2566	72
19	·3316	·330	·3256	·3443	2·9042	·9455	1·161	1·2392	71
20	·3491	·347	·3420	·3640	2·7475	·9397	1·147	1·2217	70
21	·3665	·364	·3584	·3839	2·6051	·9336	1·133	1·2043	69
22	·3840	·382	·3746	·4040	2·4751	·9272	1·118	1·1868	68
23	·4014	·399	·3907	·4245	2·3559	·9205	1·104	1·1694	67
24	·4189	·416	·4067	·4452	2·2460	·9135	1·089	1·1519	66

25	·4363	·433	·4226	·4663	2·1445	·9063	1·075	1·1345	65
26	·4538	·450	·4384	·4877	2·0503	·8988	1·060	1·1170	64
27	·4712	·467	·4540	·5095	1·9626	·8910	1·045	1·0996	63
28	·4887	·484	·4695	·5317	1·8807	·8829	1·030	1·0821	62
29	·5061	·501	·4848	·5543	1·8040	·8746	1·015	1·0647	61
30	·5236	·518	·5000	·5774	1·7321	·8660	1·000	1·0472	60
31	·5411	·534	·5150	·6009	1·6643	·8572	·985	1·0297	59
32	·5585	·551	·5299	·6249	1·6003	·8480	·970	1·0123	58
33	·5760	·568	·5446	·6494	1·5399	·8387	·954	·9948	57
34	·5934	·585	·5592	·6745	1·4826	·8290	·939	·9774	56
35	·6109	·601	·5736	·7002	1·4281	·8192	·923	·9599	55
36	·6283	·618	·5878	·7265	1·3764	·8090	·908	·9425	54
37	·6458	·635	·6018	·7536	1·3270	·7986	·892	·9250	53
38	·6632	·651	·6157	·7813	1·2799	·7880	·877	·9076	52
39	·6807	·668	·6293	·8098	1·2349	·7771	·861	·8901	51
40	·6981	·684	·6428	·8391	1·1918	·7660	·845	·8727	50
41	·7156	·700	·6561	·8693	1·1504	·7547	·829	·8552	49
42	·7330	·717	·6691	·9004	1·1106	·7431	·813	·8378	48
43	·7505	·733	·6820	·9325	1·0724	·7314	·797	·8203	47
44	·7679	·749	·6947	·9657	1·0355	·7193	·781	·8029	46
45°	·7854	·765	·7071	1·0000	1·0000	·7071	·765	·7854	45°
			Cosine.	Co-tangent.	Tangent.	Sine.	Chord.	Radians.	Degrees.
								Angle.	

TABLE OF SQUARES, CUBES, ETC.

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
1	1	1	1	1	6	36	216	2·4495	1·8171
1·1	1·21	1·331	1·0488	1·0323	6·1	37·21	226·981	2·4698	1·8272
1·2	1·44	1·728	1·0955	1·0627	6·2	38·44	238·328	2·49	1·8371
1·3	1·69	2·197	1·1402	1·0914	6·3	39·69	250·047	2·51	1·8469
1·4	1·96	2·744	1·1882	1·1187	6·4	40·96	262·144	2·5298	1·8566
1·5	2·25	3·375	1·2247	1·1447	6·5	42·25	274·625	2·5495	1·8663
1·6	2·56	4·096	1·2649	1·1696	6·6	43·56	287·496	2·569	1·8758
1·7	2·89	4·913	1·3038	1·1935	6·7	44·89	300·763	2·5884	1·8852
1·8	3·24	5·832	1·3416	1·2164	6·8	46·24	314·432	2·6077	1·8945
1·9	3·61	6·859	1·3784	1·2386	6·9	47·61	328·509	2·6268	1·9038
2	4	8	1·4142	1·2599	7	49	343	2·6458	1·9129
2·1	4 41	9·261	1·4491	1·2806	7·1	50·41	357·911	2·6646	1·922
2·2	4·84	10·648	1·4832	1·3006	7·2	51·84	373·248	2·6833	1·931
2·3	5·29	12·167	1·5166	1·32	7·3	53·29	389·017	2·7019	1·9399
2·4	5·76	13·824	1·5492	1·3389	7·4	54·76	405·224	2·7203	1·9487
2·5	6·25	15·625	1·5811	1·3572	7·5	56·25	421·875	2·7386	1·9574
2·6	6·76	17·576	1·6125	1·3751	7·6	57·76	438·976	2·7568	1·9661
2·7	7·29	19·683	1·6432	1·3925	7·7	59·29	456·533	2·7749	1·9747
2·8	7·84	21·952	1·6733	1·4095	7·8	60·84	474·552	2·7928	1·9832
2·9	8·41	24·389	1·7029	1·426	7·9	62·41	493·039	2·8107	1·9916
3	9	27	1·7321	1·4422	8	64	512	2·8284	2
3·1	9·61	29·791	1·7607	1·4581	8·1	65·61	531·441	2·846	2·0083
3·2	10·24	32·768	1·7889	1·4736	8·2	67·24	551·368	2·8636	2·0165
3·3	10·89	35·937	1·8166	1·4888	8·3	68·89	571·787	2·881	2·0247
3·4	11·56	39·304	1·8439	1·5037	8·4	70·56	592·704	2·8983	2·0328
3·5	12·25	42·875	1·8708	1·5183	8·5	73·25	614·125	2·9155	2·0408
3·6	12·96	46·656	1·8974	1·5326	8·6	73·96	636·056	2·9326	2·0488
3·7	13·69	50·653	1·9235	1·5467	8·7	75·69	658·503	2·9496	2·0567
3·8	14·44	54·872	1·9494	1·5605	8·8	77·44	681·472	2·9665	2·0646
3·9	15·21	59·319	1·9748	1·5741	8·9	79·21	704·969	2·9833	2·0724
4	16	64	2	1·5874	9	81	729	3	2·0801
4·1	16·81	68·921	2·0249	1·6005	9·1	82·81	753·571	3·0166	2·0878
4·2	17·64	74·088	2·0494	1·6134	9·2	84·64	778·688	3·0332	2·0954
4·3	18·49	79·507	2·0736	1·6261	9·3	86·49	804·357	3·0496	2·1029
4·4	19·36	85·184	2·0976	1·6386	9·4	88·36	830·584	3·0659	2·1105
4·5	20·25	91·125	2·1213	1·651	9·5	90·25	857·375	3·0822	2·1179
4·6	21·16	97·336	2·1448	1·6631	9·6	92·16	884·736	3·0984	2·1253
4·7	22·09	103·823	2·1680	1·6751	9·7	94·09	912·673	3·1145	2·1327
4·8	23·04	110·592	2·1909	1·6869	9·8	96·04	941·192	3·1305	2·14
4·9	24·01	117·649	2·2136	1·6985	9·9	98·01	970·299	3·1464	2·1472
5	25	125	2·2361	1·71	10	100	1000	3·1623	2·1544
5·1	26·01	132·651	2·2583	1·7213	10·1	102·01	1030·301	3·178	2·1616
5·2	27·04	140·608	2·2804	1·7325	10·2	104·04	1061·208	3·1937	2·1687
5·3	28·09	148·877	2·3022	1·7435	10·3	106·09	1092·727	3·2094	2·1757
5·4	29·16	157·464	2·3238	1·7544	10·4	108·16	1124·863	3·2249	2·1828
5·5	30·25	166·375	2·3452	1·7652	10·5	110·25	1157·625	3·2404	2·1897
5·6	31·36	175·616	2·3664	1·7758	10·6	112·36	1191·016	3·2558	2·1967
5·7	32·49	185·193	2·3875	1·7863	10·7	114·49	1225·043	3·2711	2·2036
5·8	33·64	195·112	2·4083	1·7967	10·8	116·64	1259·712	3·2863	2·2104
5·9	34·81	205·379	2·429	1·807	10·9	118·81	1295·029	3·3015	2·2172

TABLE OF SQUARES, ETC.—*Continued.*

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
11	121	1331	3·3166	2·2239	16	256	4096	4	2·5198
11·1	123·21	1367·631	3·3317	2·2307	16·1	259·21	4173·281	4·0125	2·5251
11·2	125·44	1406·928	3·3466	2·2374	16·2	262·44	4251·528	4·0249	2·5303
11·3	127·69	1442·897	3·3615	2·2441	16·3	265·69	4330·747	4·0373	2·5355
11·4	129·96	1481·544	3·3764	2·2506	16·4	268·96	4410·944	4·0497	2·5407
11·5	132·25	1520·875	3·3912	2·2572	16·5	272·25	4492·125	4·062	2·5458
11·6	134·56	1560·896	3·4059	2·2637	16·6	275·56	4574·296	4·0743	2·5509
11·7	136·89	1601·613	3·4205	2·2702	16·7	278·89	4657·463	4·0866	2·5561
11·8	139·24	1643·032	3·4351	2·2766	16·8	282·24	4741·632	4·0988	2·5612
11·9	141·61	1685·159	3·4496	2·2831	16·9	285·61	4826·809	4·111	2·5662
12	144	1728	3·4641	2·2894	17	289	4913	4·1231	2·571
12·1	146·41	1771·561	3·4785	2·2957	17·1	292·41	5000·211	4·1352	2·5763
12·2	148·84	1815·848	3·4928	2·3021	17·2	295·84	5088·448	4·1473	2·5813
12·3	151·29	1860·867	3·5071	2·3084	17·3	299·29	5177·717	4·1593	2·5863
12·4	153·76	1906·624	3·5214	2·3146	17·4	302·76	5268·024	4·1713	2·5913
12·5	156·25	1953·125	3·5355	2·3298	17·5	306·25	5359·375	4·1833	2·5962
12·5	158·76	2000·376	3·5496	2·327	17·6	309·76	5451·776	4·1952	2·6012
12·7	161·29	2048·383	3·5637	2·3331	17·7	313·29	5545·233	4·2071	2·6061
12·8	163·84	2097·152	3·5777	2·3391	17·8	316·84	5639·752	4·219	2·611
12·9	166·41	2146·689	3·5917	2·3453	17·9	320·41	5735·339	4·2308	2·6159
13	169	2197	3·6056	2·3513	18	324	5832	4·2426	2·6207
13·1	171·61	2248·091	3·6194	2·3573	18·1	327·61	5929·741	4·2544	2·6256
13·2	174·24	2299·968	3·6332	2·3633	18·2	331·24	6028·568	4·2661	2·6304
13·3	176·89	2352·637	3·6469	2·3693	18·3	334·89	6128·487	4·2778	2·6352
13·4	179·56	2406·104	3·6606	2·3752	18·4	338·56	6229·504	4·2895	2·64
13·5	182·25	2460·375	3·6742	2·3811	18·5	342·25	6331·625	4·3012	2·6448
13·6	184·96	2515·456	3·6878	2·387	18·6	345·96	6434·856	4·3128	2·6495
13·7	187·69	2571·353	3·7013	2·3928	18·7	349·69	6539·203	4·3248	2·6543
13·8	190·44	2628·072	3·7148	2·3986	18·8	353·44	6644·672	4·3359	2·659
13·9	193·21	2685·619	3·7283	2·4044	18·9	357·21	6751·269	4·3474	2·6637
14	196	2744	3·7417	2·4101	19	361	6859	4·3589	2·6684
14·1	198·81	2803·221	3·755	2·4159	19·1	364·81	6967·871	4·3704	2·6731
14·2	201·64	2863·288	3·7683	2·4216	19·2	368·64	7077·888	4·3818	2·6777
14·3	204·49	2924·207	3·7815	2·4272	19·3	372·49	7189·057	4·3932	2·6824
14·4	207·36	2985·984	3·7947	2·4329	19·4	376·36	7301·384	4·4045	2·6870
14·5	210·25	3048·625	3·8079	2·4385	19·5	380·25	7414·875	4·4159	2·6916
14·6	213·16	3112·136	3·821	2·4441	19·6	384·16	7529·536	4·4272	2·6962
14·7	216·09	3176·523	3·8341	2·4497	19·7	388·09	7645·373	4·4385	2·7008
14·8	219·04	3241·792	3·8471	2·4552	19·8	392·04	7762·392	4·4497	2·7053
14·9	222·01	3307·949	3·86	2·4607	19·9	396·01	7880·599	4·4609	2·7099
15	225	3375	3·873	2·4662	20	400	8000	4·4721	2·7144
15·1	228·01	3442·951	3·8859	2·4717	20·1	404·01	8120·601	4·4833	2·7189
15·2	231·04	3511·808	3·8987	2·4771	20·2	408·04	8242·408	4·4944	2·7234
15·3	234·09	3581·577	3·9115	2·4825	20·3	412·09	8365·427	4·5055	2·7279
15·4	237·16	3652·264	3·9243	2·4879	20·4	416·16	8489·664	4·5166	2·7324
15·5	240·25	3723·875	3·937	2·4933	20·5	420·25	8615·125	4·5277	2·7368
15·6	243·36	3796·416	3·9497	2·4987	20·6	424·36	8741·816	4·5387	2·7413
15·7	246·49	3869·893	3·9623	2·504	20·7	428·49	8869·743	4·5497	2·7457
15·8	249·64	3944·312	3·9749	2·5093	20·8	432·64	8998·912	4·5607	2·7502
15·9	252·81	4019·679	3·9875	2·5146	20·9	436·81	9129·329	4·5716	2·7545

TABLE OF SQUARES, ETC.—*Continued.*

No.	Square.	Cube.	Square Root.	Cube Root.
21	441	9261	4·5826	2·7589
21·1	445·21	9393·931	4·5935	2·7633
21·2	449·44	9528·128	4·6043	2·7676
21·3	453·69	9663·597	4·6152	2·772
21·4	457·96	9800·344	4·626	2·7763
21·5	462·25	9938·375	4·6368	2·7806
21·6	466·56	10077·696	4·6476	2·7849
21·7	470·89	10218·313	4·6586	2·7893
21·8	475·24	10360·232	4·669	2·7935
21·9	479·61	10503·459	4·6797	2·7978
22	484	10648	4·6904	2·8021
22·1	488·41	10793·861	4·7011	2·8063
22·2	492·84	10941·048	4·7117	2-8105
22·3	497·29	11089·567	4·7223	2·8147
22·4	501·76	11239·424	4·7329	2·8189
22·5	506·25	11390·625	4·7434	2·8231
22·6	510·76	11543·176	4·7539	2·8273
22·7	515·29	11697·083	4·7644	2·8314
22·8	519·84	11852·352	4·7749	2·8356
22·9	524·41	12008·989	4·7854	2·8397
23	529	12167	4·7958	2·8438
23·1	533·61	12326·391	4·8062	2·8479
23·2	538·24	12487·168	4·8166	2·8521
23·3	542·89	12649·337	4·827	2·8562
23·4	547·56	12812·904	4·8373	2·8603
23·5	552·25	12977·875	4·8477	2·8643
23·6	556·96	13144·256	4·858	2·8684
23·7	561·69	13312·053	4·8683	2·8724
23·8	566·44	13481·272	4·8785	2·8765
23·9	571·21	13651·919	4·8888	2·8805
24	576	13824	4·899	2·8845
24·1	580·81	13997·521	4·9092	2·8885
24·2	585·64	14172·488	4·9193	2·8925
24·3	590·49	14348·907	4·9295	2·8965
24·4	595·36	14526·784	4·9396	2·9004
24·5	600·25	14706·125	4·9497	2·9044
24·6	605·16	14886·936	4·9598	2·9083
24·7	610·09	15069·223	4·9699	2·9123
24·8	615·04	15252·992	4·9799	2·9162
24·9	620·01	15438·249	4·9899	2·9201
25	625	15625	5	2·9240
25·1	630·01	15813·251	5·01	2·9279
25·2	635·04	16003·008	5·02	2·9318
25·3	640·09	16194·277	5·0299	2·9357
25·4	645·16	16387·064	5·0398	2·9395
25·5	650·25	16581·375	5·0498	2·9434
25·6	655·36	16777·216	5·0596	2·9472
25·7	660·49	16974·593	5·0695	2·9511
25·8	665·64	17173·512	5·0794	2·9549
25·9	670·81	17373·979	5·0892	2·9587
26	676	17576	5·099	2·9625
26·1	681·21	17779·581	5·1088	2·9663
26·2	686·44	17984·728	5·1186	2·9701
26·3	691·69	18191·447	5·1284	2·9738
26·4	696·96	18399·744	5·1381	2·9776
26·5	702·25	18609·625	5·1478	2·9814
26·6	707·56	18821·096	5·1575	2·9851
26·7	712·89	19034·163	5·1672	2·9888
26·8	718·24	19248·832	5·1769	2·9926
26·9	723·61	19465·109	5·1865	2·9963
27	729	19683	5·1962	3
27·1	734·41	19902·511	5·2058	3·0037
27·2	739·84	20123·648	5·2154	3·0074
27·3	745·29	20346·417	5·2249	3·0111
27·4	750·76	20570·824	5·2345	3·0147
27·5	756·25	20796·875	5·244	3·0184
27·6	761·76	21024·576	5·2536	3·0221
27·7	767·29	21353·933	5·2631	3·0257
27·8	772·84	21484·952	5·2726	3·0293
27·9	778·41	21717·639	5·282	3·033
28	784	21952	5·2915	3·0366
28·1	789·61	22188·041	5·3509	3·0402
28·2	795·24	22425·768	5·3104	3·0438
28·3	800·89	22665·187	5·3198	3·0474
28·4	806·56	22906·304	5·3292	3·051
28·5	812·26	23149·125	5·3385	3·0546
28·6	817·96	23393·656	5·3479	3·0581
28·7	823·69	23639·903	5·3572	3·0617
28·8	829·44	23887·872	5·3666	3·0652
28·9	835·21	24137·569	5·3759	3·0688
29	841	24389	5·3852	3·0723
29·1	846·81	24642·171	5·3944	3·0758
29·2	852·64	24897·088	5·4037	3·0794
29·3	858·49	25153·757	5·4129	3·0829
29·3	864·36	25412·184	5·4222	3·0864
29·5	870·25	25672·375	5·4314	3·0899
29·6	876·16	25934·336	5·4406	3·0934
29·7	882·09	26198·073	5·4498	3·0968
29·8	888·04	26463·592	5·4589	3·1003
29·9	894·01	26730·899	5·4681	3·1038
30	900	27000	5·4772	3·1072
30·1	906·01	27270·901	5·4863	3·1107
30·2	912·04	27543·608	5·4954	3·1141
30·3	918·09	27818·127	5·5045	3·1176
30·4	924·16	28094·464	5·5136	3·121
30·5	930·25	28372·623	5·5226	5·1244
30·6	936·36	28652·616	5·5317	3·1278
30·7	942·49	28934·443	5·5407	3·1312
30·8	948·64	29218·112	5·5497	3·1346
30·9	954·81	29503·629	5·5587	3·138

Table of Squares, Etc.—*Continued.*

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
31	961	29791	5·5678	3·1414	36	1296	46656	6	3·3019
31·1	967·21	30080·231	5·5767	3·1448	36·1	1303·21	47045·881	6·0083	3·305
31·2	973·44	30371·328	5·5857	3·1481	36·2	1310·44	47437·928	6·0166	3·308
31·3	979·69	30664·297	5·5946	3·1515	36·3	1317·69	47832·147	6·0249	3·3111
31·4	985·96	30959·144	5·6035	3·1548	36·4	1324·96	48228·544	6·0332	3·3141
31·5	992·25	31255·875	5·6124	3·1582	36·5	1332·25	48627·125	6·0415	3·3171
31·6	998·56	31554·496	5·6213	3·1615	36·6	1339·56	49027·896	6·0498	3·3202
31·7	1004·89	31855·013	5·6302	3·1648	36·7	1346·89	49430·863	6·0581	3·3232
31·8	1011·24	32157·432	5·6391	3·1671	36·8	1354·24	49836·032	6·0663	3·3262
31·9	1017·61	32461·759	5·648	3·1715	36·9	1361·61	50243·409	6·0745	3·3292
32	1024	32768	5·6569	3·1748	37	1369	50653	6·0828	3·3322
32·1	1030·41	33076·161	5·6656	3·1781	37·1	1376·41	51064·811	6·091	3·3352
32·2	1036·84	33386·248	5·6745	3·1814	37·2	1383·84	51478·848	6·0992	3·3382
32·3	1043·29	33698·267	5·6833	3·1847	37·3	1391·29	51895·117	6·1074	3·3412
32·4	1049·76	34012·224	5·6921	3·188	37·4	1398·76	52313·624	6·1156	3·3442
32·5	1056·25	34328·125	5·7008	3·1913	37·5	1406·25	52734·375	6·1237	3·3472
32·6	1062·76	34645·976	5·7096	3·1945	37·6	1413·76	53157·376	6·1319	3·3501
32·7	1069·29	34965·783	5·7183	3·1978	37·7	1421·29	53582·633	6·14	3·3531
32·8	1075·84	35287·552	5·7271	3·201	37·8	1428·84	54010·152	6·1482	3·3561
32·9	1082·41	35611·289	5·7358	3·2043	37·9	1436·41	54439·939	6·1563	3·359
33	1089	35937	5·7446	3·2075	38	1444	54872	6·1644	3·362
33·1	1095·61	36264·691	5·7532	3·2108	38·1	1451·61	55306·341	6·1725	3·3649
33·2	1102·24	36594·368	5·7619	3·214	38·2	1459·24	55742·968	6·1806	3·3679
33·3	1108·89	36926·037	5·7706	3·2172	38·3	1466·89	56181·887	6·1887	3·3708
33·4	1115·56	37259·704	5·7792	3·2204	38·4	1474·56	56623·104	6·1968	3·3737
33·5	1132·25	37595·375	5·7879	3·2237	38·5	1482·25	57066·625	6·2048	3·3767
33·6	1128·96	37933·056	5·7965	3·2269	38·6	1489·96	57512·456	6·2129	3·3796
33·7	1135·69	38272·753	5·8051	3·2301	38·7	1497·69	57960·603	6·2209	3·3825
33·8	1142·44	38614·472	5·8137	3·2332	38·8	1505·44	58411·072	6·229	3·3854
33·9	1149·21	38958·219	5·8223	3·2364	38·9	1513·21	58863·869	6·237	3·3883
34	1156	39304	5·831	3·2396	39	1521	59319	6·245	3·3912
34·1	1162·81	39651·821	5·8395	3·2428	39·1	1528·81	59776·471	6·253	3·3941
34·2	1169·64	40001·688	5·848	3·246	39·2	1536·64	60236·288	6·261	3·397
34·3	1176·49	40353·607	5·8566	3·2491	39·3	1544·49	60698·457	6·269	3·3999
34·4	1183·36	40707·584	5·8651	3·2522	39·4	1552·36	61162·984	6·2769	3·4028
34·5	1190·25	41063·625	5·8736	3·2554	39·5	1560·25	61629·875	6·2849	3·4056
34·6	1197·16	41421·736	5·8821	3·2586	39·6	1568·16	62099·136	6·2929	3·4085
34·7	1204·09	41781·923	5·8906	3·2617	39·7	1576·09	62570·773	6·3008	3·4114
34·8	1211·04	42144·192	5·8991	3·2648	39·8	1584·04	63044·792	6·3087	3·4142
34·9	1218·01	42508·549	5·9076	3·2679	39·9	1592·01	63521·199	6·3166	3·4171
35	1225	42875	5·9161	3·2711	40	1600	64000	6·3246	3·42
35·1	1232·01	43243·551	5·9245	3·2742	40·1	1608·01	64481·201	6·3325	3·4228
35·2	1239·04	43614·208	5·933	3·2773	40·2	1616·04	64964·808	6·3404	3·4256
35·3	1246·09	43986·977	5·9414	3·2804	40·3	1624·09	65450·827	6·3482	3·4285
35·4	1253·16	44361·864	5·9498	3·2835	40·4	1632·16	65939·264	6·3561	3·4313
35·5	1260·25	44738·875	5·9582	3·2866	40·5	1640·25	66430·125	6·3639	3·4341
35·6	1267·36	45118·016	5·9666	3·2897	40·6	1648·36	66923·416	6·3718	3·437
35·7	1274·49	45499·293	5·9749	3·2927	40·7	1656·49	67419·143	6·3796	3·4398
35·8	1281·64	45882·712	5·9833	3·2958	40·8	1664·64	67911·312	6·3875	3·4426
35·9	1288·81	46268·279	5·9917	3·2989	40·9	1672·81	68417·929	6·3953	3·4454

TABLE OF SQUARES, ETC.—*Continued.*

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
41	1681	68921	6·4031	3·4482	46	2116	97336	6·7823	3·583
41·1	1689·21	69426·531	6·4109	3·451	46·1	2125·21	97972·181	6·7897	3·5856
41·2	1697·44	69934·528	6·4187	3·4538	46·2	2134·44	98611·128	6·7971	3·5882
41·3	1705·69	70444·997	6·4265	3·4566	46·3	2143·69	99252·847	6·8044	3·5908
41·4	1713·96	70957·944	6·4343	3·4594	46·4	2152·96	99897·344	6·8118	3·5934
41·5	1722·25	71473·375	6·4421	3·4622	46·5	2162·25	100544·625	6·8191	3·596
41·6	1730·56	71991·296	6·4498	3·465	46·6	2171·56	101194·696	6·8264	3·5986
41·7	1738·89	72511·713	6·4575	3·4677	46·7	2180·89	101847·563	6·8337	3·6011
41·8	1747·24	73034·632	6·4653	3·4705	46·8	2190·24	102503·232	6·8411	3·6037
41·9	1755·61	73560·059	6·473	3·4733	46·9	2199·61	103161·709	6·8484	3·6063
42	1764	74088	6·4807	3·476	47	2209	103823	6·8557	3·6088
42·1	1772·41	74618·461	6·4884	3·4788	47·1	2218·41	104487·111	6·8629	3·6114
42·2	1780·84	75151·448	6·4961	3·4815	47·2	2227·84	105154·048	6·8702	3·6139
42·3	1789·29	75686·967	6·5038	3·4843	47·3	2237·29	105823·817	6·8775	3·6165
42·4	1797·76	76225·024	6·5115	3·487	47·4	2246·76	106496·424	6·8848	3·619
42·5	1806·25	76765·625	6·5192	3·4898	47·5	2256·25	107171·875	6·892	3·6216
42·6	1814·76	77308·776	6·5268	3·4915	47·6	2265·76	107850·176	6·8993	3·6241
42·7	1823·29	77854·483	6·5345	3·4952	47·7	2275·29	108531·333	6·9065	3·6267
42·8	1831·84	78402·752	6·5422	3·498	47·8	2284·84	109215·352	6·9138	3·6292
42·9	1840·41	78953·589	6·5498	3·5007	47·9	2294·41	109902·239	6·921	3·6317
43	1849	79507	6·5574	3·5034	48	2304	110592	6·9282	3·6342
43·1	1857·61	80062·991	6·5651	3·5061	48·1	2313·61	111284·641	6·9354	3·6368
43·2	1866·24	80621·568	6·5727	3·5088	48·2	2323·24	111980·168	6·9426	3·6393
43·3	1874·89	81182·737	6·5803	3·5115	48·3	2332·89	112678·587	6·9498	3·6418
43·4	1883·56	81746·504	6·5879	3·5142	48·4	2342·56	113379·904	6·957	3·6443
43·5	1892·25	82312·875	6·5954	3·5169	48·5	2352·25	114084·125	6·9642	3·6468
43·6	1900·96	82881·856	6·603	3·5196	48·6	2361·96	114791·256	6·9714	3·6493
43·7	1909·69	83453·453	6·6106	3·5223	48·7	2371·69	115501·303	6·9785	3·6513
43·8	1918·44	84027·672	6·6182	3·525	48·8	2381·44	116214·272	6·9857	3·6543
43·9	1927·21	84604·519	6·6257	3·5277	48·9	2391·21	116930·169	6·9929	3·6568
44	1936	85184	6·6332	3·5303	49	2401	117649	7	3·6593
44·1	1944·81	85766·121	6·6408	3·533	49·1	2410·81	118370·771	7·0071	3·6618
44·2	1953·64	86350·888	6·6483	3·5357	49·2	2420·64	119095·488	7·0143	3·6643
44·3	1962·49	86938·307	6·6558	3·5384	49·3	2430·49	119823·157	7·0214	3·6668
44·4	1971·36	87528·384	6·6633	3·541	49·4	2440·36	120553·784	7·0285	3·6692
44·5	1980·25	88121·125	6·6708	3·5437	49·5	2450·25	121287·375	7·0356	3·6717
44·6	1989·16	88716·536	6·6783	3·5463	49·6	2460·16	122023·936	7·0427	3·6742
44·7	1998·09	89314·623	6·6858	3·549	49·7	2470·09	122763·473	7·0498	3·6766
44·8	2007·04	89915·392	6·6933	3·5516	49·8	2480·04	123505·992	7·0569	3·6791
44·9	2016·01	90518·849	6·7007	3·5543	49·9	2490·01	124251·499	7·064	3·6816
45	2025	91125	6·7082	3·5569	50	2500	125000	7·0711	3·684
45·1	2034·01	91733·851	6·7157	3·5595	51	2601	132651	7·1414	3·7084
45·2	2043·04	92345·408	6·7231	3·5622	52	2704	140608	7·2111	3·7325
45·3	2052·90	92959·677	6·7305	3·5648	53	2809	148877	7·2801	3·7563
45·4	2061·16	93576·664	6·738	3·5674	54	2916	157464	7·3485	3·7798
45·5	2070·25	94196·375	6·7454	3·57	55	3025	166375	7·4162	3·803
45·6	2079·36	94818·816	6·7528	3·5726	56	3136	175616	7·4833	3·8259
45·7	2088·49	95443·993	6·7602	3·5752	57	3249	185193	7·5498	3·8485
45·8	2097·64	96071·912	6·7676	3·5778	58	3364	195112	7·6158	3·8709
45·9	2106·81	96702·579	6·775	3·5805	59	3481	205379	7·6811	3·893

TABLE OF SQUARES, ETC.—*Continued.*

No.	Square.	Cube.	Square Root.	Cube Root.	No.	Square.	Cube.	Square Root.	Cube Root.
60	3600	216000	7·746	3·9149	80	6400	512000	8·9443	4·3089
61	3721	226981	7·8102	3·9365	81	6561	531441	9	4·3267
62	3844	238328	7·874	3·9579	82	6724	551368	9·0554	4·3445
63	3969	250047	7·9373	3·9791	83	6889	571787	9·1104	4·3621
64	4096	262144	8	4	84	7056	592704	9·1652	4·3795
65	4225	274625	8·0623	4·0207	85	7225	614125	9·2195	4·3968
66	4356	287496	8·124	4·0412	86	7396	636056	9·2736	4·414
67	4489	300763	8·1854	4·0615	87	7569	658503	9·3274	4·431
68	4624	314432	8·2462	4·0817	88	7744	681472	9·3808	4·448
69	4761	328509	8·3066	4·1016	89	7921	704969	9·434	4·4647
70	4900	343000	8·3666	4·1213	90	8100	729000	9·4868	4·4814
71	5041	357911	8·4261	4·1408	91	8281	753571	9·5394	4·4979
72	5184	373248	8·4853	4·1602	92	8464	778688	9·5917	4·5144
73	5329	389017	8·544	4·1793	93	8649	804357	9·6437	4·5307
74	5476	405224	8·6023	4·1983	94	8836	830584	9·6954	4·5468
75	5625	421875	8·6603	4·2172	95	9025	857375	9·7468	4·5629
76	5776	438976	8·7178	4·2358	96	9216	884736	9·798	4·5789
77	5929	456533	8·775	4·2543	97	9409	912673	9·8489	4·5947
78	6084	474552	8·8318	4·2727	98	9604	941192	9·8995	4·6104
79	6241	493039	8·8882	4·2908	99	9801	970299	9·9499	4·6261

AREAS AND CIRCUMFERENCES OF CIRCLES UP TO 6 IN. DIAMETER (ADVANCING BY 32NDS AND 16THS).

Dia.	Circum.	Area.	Dia.	Circum.	Area.	Dia.	Circum.	Area.
$\frac{1}{32}$	·0981	·00077	$1\frac{3}{8}$	4·3197	1·4848	$3\frac{11}{16}$	11·584	10·679
$\frac{1}{16}$	·1963	·00307	$1\frac{7}{16}$	4·516	1·6229	$3\frac{3}{4}$	11·781	11·044
$\frac{3}{32}$	·2945	·0069	$1\frac{1}{2}$	4·7124	1·7671	$3\frac{13}{16}$	11·977	11·416
$\frac{1}{8}$	·3927	·01227	$1\frac{9}{16}$	4·9087	1·9175	$3\frac{7}{8}$	12·173	11·793
$\frac{5}{32}$	·4908	·0192	$1\frac{5}{8}$	5·1051	2·0739	$3\frac{15}{16}$	12·369	12·177
$\frac{3}{16}$	·589	·02761	$1\frac{11}{16}$	5·3014	2·2365	4	12·566	12·566
$\frac{7}{32}$	·6872	·0376	$1\frac{3}{4}$	5·4978	2·4052	$4\frac{1}{16}$	12·762	12·962
$\frac{1}{4}$	·7854	·04909	$1\frac{13}{16}$	5·6941	2·58	$4\frac{1}{8}$	12·959	13·364
$\frac{9}{32}$	·8835	·0621	$1\frac{7}{8}$	5·8905	2·7611	$4\frac{3}{16}$	13·155	13·772
$\frac{5}{16}$	·9817	·0767	$1\frac{15}{16}$	6·0868	2·9483	$4\frac{1}{4}$	13·351	14·186
$\frac{11}{32}$	1·0799	·0928	2	6·2832	3·1416	$4\frac{5}{16}$	13·547	14·606
$\frac{3}{8}$	1·1781	·1104	$2\frac{1}{16}$	6·4795	3·3410	$4\frac{3}{8}$	13·744	15·033
$\frac{13}{32}$	1·2762	·1296	$2\frac{1}{8}$	6·6759	3·5465	$4\frac{7}{16}$	13·94	15·465
$\frac{7}{16}$	1·3744	·1503	$2\frac{3}{16}$	6·8722	3·7584	$4\frac{1}{2}$	14·137	15·904
$\frac{15}{32}$	1·4726	·1725	$2\frac{1}{4}$	7·0686	3·976	$4\frac{9}{16}$	14·333	16·349
$\frac{1}{2}$	1·5708	·1963	$2\frac{5}{16}$	7·2649	4·2	$4\frac{5}{8}$	14·529	16·8
$\frac{17}{32}$	1·6689	·2216	$2\frac{3}{8}$	7·4613	4·4302	$4\frac{11}{16}$	14·725	17·257
$\frac{9}{16}$	1·7771	·2485	$2\frac{7}{16}$	7·6576	4·6664	$4\frac{3}{4}$	14·922	17·72
$\frac{19}{32}$	1·8653	·2768	$2\frac{1}{2}$	7·854	4·9087	$4\frac{13}{16}$	15·119	18·19
$\frac{5}{8}$	1·9635	·3068	$2\frac{9}{16}$	8·0503	5·1573	$4\frac{7}{8}$	15·315	18·665
$\frac{21}{32}$	2·0616	·3382	$2\frac{5}{8}$	8·2467	5·4119	$4\frac{15}{16}$	15·511	19·147
$\frac{11}{16}$	2·1598	·3712	$2\frac{11}{16}$	8·443	5·6723	5	15·708	19·635
$\frac{23}{32}$	2·258	·4057	$2\frac{3}{4}$	8·6394	5·9395	$5\frac{1}{16}$	15·904	20·129
$\frac{3}{4}$	2·3562	·4417	$2\frac{13}{16}$	8·8357	6·2126	$5\frac{1}{8}$	16·1	20·629
$\frac{25}{32}$	2·4543	·4793	$2\frac{7}{8}$	9·0321	6·4918	$5\frac{3}{16}$	16·296	21·135
$\frac{13}{16}$	2·5525	·5185	$2\frac{15}{16}$	9·2284	6·7772	$5\frac{1}{4}$	16·493	21·647
$\frac{27}{32}$	2·6507	·5591	3	9·4248	7·0686	$5\frac{5}{16}$	16·689	22·166
$\frac{7}{8}$	2·7489	·6013	$3\frac{1}{16}$	9·6211	7·3662	$5\frac{3}{8}$	16·886	22·69
$\frac{29}{32}$	2·847	·645	$3\frac{1}{8}$	9·8175	7·6699	$5\frac{7}{16}$	17·082	23·221
$\frac{15}{16}$	2·9452	·6903	$3\frac{3}{16}$	10·014	7·9798	$5\frac{1}{2}$	17·278	23·758
$\frac{31}{32}$	3·0434	·737	$3\frac{1}{4}$	10·21	8·2957	$5\frac{9}{16}$	17·474	24·301
1	3·1416	·7854	$3\frac{5}{16}$	10·406	8·618	$5\frac{5}{8}$	17·671	24·85
$1\frac{1}{16}$	3·3379	·8866	$3\frac{3}{8}$	10·602	8·9462	$5\frac{11}{16}$	17·867	25·406
$1\frac{1}{8}$	3·5343	·994	$3\frac{7}{16}$	10·799	9·2807	$5\frac{3}{4}$	18·064	25·967
$1\frac{3}{16}$	3·7306	1·1075	$3\frac{1}{2}$	10·995	9·6211	$5\frac{13}{16}$	18·261	26·535
$1\frac{1}{4}$	3·927	1·2271	$3\frac{9}{16}$	11·191	9·968	$5\frac{7}{8}$	18·457	27·108
$1\frac{5}{16}$	4·1233	1·353	$3\frac{5}{8}$	11·388	10·32	$5\frac{15}{16}$	18·653	27·688

AREAS OF CIRCLES ADVANCING BY 10THS.

Diam.	Areas.										Diam.
	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	
0	·0	·0078	·0314	·0706	·1256	·1963	·2827	·3848	·5026	·6361	0
1	·7854	·9503	1·1309	1·3273	1·5393	1·7671	2·0106	2·2698	2·5446	2·8352	1
2	3·1416	3·4636	3·8013	4·1547	4·5239	4·9087	5·3093	5·7255	6·1575	6·6052	2
3	7·0686	7·5476	8·0424	8·5530	9·0792	9·6211	10·1787	10·7521	11·3411	11·9459	3
4	12·5664	13·2025	13·8544	14·5220	15·2053	15·9043	16·6190	17·3494	18·0951	18·8574	4
5	19·6350	20·4282	21·2372	22·0618	22·9022	23·7583	24·6301	25·5176	26·4208	27·3397	5
6	28·2744	29·2247	30·1907	31·1725	32·1699	33·1831	34·2120	35·2566	36·3168	37·3928	6
7	38·4846	39·5920	40 7151	41·8539	43·0085	44·1787	45·3647	46·5663	47·7837	49·0168	7
8	50·2656	51·5300	52·8102	54·1062	55·4178	56·5471	58·0381	59·4469	60·8213	62·2115	8
9	63·6174	65·0389	66·4764	67·9292	69·3979	70·8823	72·3824	73·8982	75·4298	76·9770	9
10	78·5400	80·1186	81·7130	83·3230	84·9488	86·5903	88·2475	89·9204	91·6090	93·3133	10
11	95·0334	96·7691	98·5205	100·287	102·070	103·869	105·683	107·513	109·359	111·220	11
12	113·097	114·990	116·898	118·823	120·763	122·718	124·690	126·677	128·679	130·698	12
13	132·732	134·782	136·848	138·929	141·026	143·139	145·267	147·411	149·571	151·747	13
14	153·938	156·145	158·368	160·606	162·860	165·130	167·415	169·717	172·034	174·366	14
15	176·715	179·079	181·458	183·854	186·265	188·692	191·134	193·593	196·067	198·556	15
16	201·062	203·583	206·120	208·672	211·241	213·825	216·424	219·040	221·671	224·318	16
17	226·980	229·658	232·352	235·062	237·787	240·528	243·285	246·057	248·846	251·650	17
18	254·469	257·304	260·125	263·022	265·905	268·803	271·716	274·646	277·591	280·552	18
19	283·529	286·521	289·529	292·553	295·598	298·648	301·719	304·805	307·908	311·026	19
20	314·160	317·309	320·474	323·655	326·852	330·064	333·292	336·536	339·795	343·070	20
21	346·361	349·667	352·990	356·328	359·681	363·051	366·436	369·837	373·253	376·685	21
22	380·133	383·597	387·076	390·571	394·082	397·608	401·150	404·708	408·282	411·871	22
23	415·476	419·097	422·733	426·385	430·053	433·737	437·436	441·151	444·881	448·628	23
24	452·390	456·168	459·961	463·770	467·595	471·436	475·292	479·164	483·052	486·955	24
25	490·875	494·809	498·760	502·726	506·708	510·706	514·719	518·748	522·793	526·854	25

Metric Equivalents.

1 centimetre . . . = 0·3937 in.
1 sq. centimetre . . . = 0·1550 sq. in.
1 cub. centimetre . . = 0·0610 cub. in.
1 kilogram . . . = 2·205 lb.
1 kilogram-metre . . = 86·82 in.-lb.
1 centimetre per sec. . . = 0·0328 ft. per sec.
1 gram per sq. centimetre . = 0·0142 lb. per sq. in.
1 gram per cub. centimetre = 62·42 lb. per cub. ft.
Acceleration due to gravity = 32·2 ft. per sec.
„ „ = 981·4 centimetres per sec.
1 in. . . . . . = 2·540 centimetres.
1 sq. in. . . . . = 6·541 sq. centimetres.
1 cub. in. . . . . = 16·38 cub. centimetres.
1 lb. (avoirdupois) . . = ·4536 kilogram.
1 ft. per sec. . . . = 30·48 centimetres per sec.
1 lb. per sq. in. . . . = 69·34 grams per sq. centimetre.
1 lb. per cub. in. . . = 27·616 grams per cub. centimetre.
1 lb. per cub. ft. . . = 0·016022 gram per cub. centimetre.
1 in.-lb. . . . . = 1152 gram-centimetres.

Equivalent Values of Millimetres and Inches.

Milli-metres.	Inches.	Milli-metres.	Inches.	Milli-metres.	Inches.	Milli-metres.	Inches.
1	·0394	27	1·0630	53	2·0866	79	3·1103
2	·0787	28	1·1024	54	2·1260	80	3·1496
3	·1181	29	1·1417	55	2·1654	81	3·1890
4	·1575	30	1·1811	56	2·2047	82	3·2284
5	·1968	31	1·2205	57	2·2441	83	3·2677
6	·2362	32	1·2598	58	2·2835	84	3·3071
7	·2756	33	1·2992	59	2·3228	85	3·3465
8	·3150	34	1·3386	60	2·3622	86	3·3859
9	·3543	35	1·3780	61	2·4016	87	3·4252
10	·3937	36	1·4173	62	2·4410	88	3·4646
11	·4331	37	1·4567	63	2·4803	89	3·5040
12	·4724	38	1·4961	64	2·5197	90	3·5433
13	·5118	39	1·5354	65	2·5591	91	3·5827
14	·5512	40	1·5748	66	2·5984	92	3·6221
15	·5906	41	1·6142	67	2·6378	93	3·6614
16	·6299	42	1·6536	68	2·6772	94	3·7008
17	·6693	43	1·6929	69	2·7166	95	3·7402
18	·7087	44	1·7323	70	2·7559	96	3·7796
19	·7480	45	1·7717	71	2·7953	97	3·8189
20	·7874	46	1·8110	72	2·8347	98	3·8583
21	·8268	47	1·8504	73	2·8740	99	3·8977
22	·8661	48	1·8898	74	2·9134	100	3·9370
23	·9055	49	1·9291	75	2·9528		
24	·9449	50	1·9685	76	2·9922	(100	mm. =
25	·9843	51	2·0079	77	3·0315	1 deci	metre.)
26	1·0236	52	2·0473	78	3·0709		

KILOGRAMMES IN POUNDS.

Kilos.	Pounds.	Kilos.	Pounds.	Kilos.	Pounds.	Kilos.	Pounds.
1	2·205	26	57·320	51	112·436	76	167·551
2	4·409	27	59·525	52	114·640	77	169·756
3	6·614	28	61·729	53	116·845	78	171·960
4	8·818	29	63·934	54	119·049	79	174·165
5	11·023	30	66·139	55	121·254	80	176·370
6	13·228	31	68·343	56	123·459	81	178·574
7	15·432	32	70·548	57	125·663	82	180·779
8	17·637	33	72·752	58	127·868	83	182·983
9	19·842	34	74·957	59	130·073	84	185·118
10	22·046	35	77·162	60	132·277	85	187·393
11	24·251	36	79·366	61	134·482	86	189·597
12	26·455	37	81·571	62	136·486	87	191·802
13	28·660	38	83·776	63	138·891	88	194·010
14	30·865	39	85·980	64	141·096	89	196·211
15	33·069	40	88·185	65	143·300	90	198·416
16	35·274	41	90·389	66	145·505	91	200·620
17	37·479	42	92·594	67	147·710	92	202·825
18	39·683	43	94·799	68	149·914	93	205·030
19	41·888	44	97·003	69	152·119	94	207·234
20	44·092	45	99·208	70	154·323	95	209·439
21	46·297	46	101·413	71	156·528	96	211·644
22	48·502	47	103·617	72	158·733	97	213·848
23	50·706	48	105·822	73	160·937	98	216·053
24	52·911	49	108·026	74	163·142	99	218·275
25	55·115	50	110·231	75	165·347	100	220·462

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